ON THE NON-MONOTONICITY OF THE DENOMINATOR OF THE SUM OF CONSECUTIVE (GENERALIZED) UNIT FRACTIONS

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Abstract

Let L_n be the least common multiple of $\{1, 2, .., n\}$ and X_n such that $\frac{X_n}{L_n} = \sum_{i=1}^{n} \frac{1}{i}$. Let $\sum_{i=a}^{b} \frac{1}{i} = \frac{u_{a,b}}{v_{a,b}}$ with $u_{a,b}$ and $v_{a,b}$ coprime. In their influential monograph [1, p. 34], Erdös and Graham ask, among many others, the following questions: Does $\gcd(X_n, L_n) > 1$ happen infinitely often? Does there, for every fixed a, exist a b such that $v_{a,b} < v_{a,b-1}$? If so, what is the least such $b = b(a)$? In this note we will investigate these issues in a more general setting, answer the first two questions in the affirmative and obtain an upper bound of $b(a) \leq 6(a-1)$ for all $a > 1$.

1. Introduction

Let $u_{a,b}$ and $v_{a,b}$ be coprime integers such that $v_{a,b}$ is positive and $\frac{u_{a,b}}{v_{a,b}} = \sum_{i=a}^{b} \frac{r_i}{is_i}$, where $\{r_i\}_{i\in\mathbb{N}}$ and $\{s_i\}_{i\in\mathbb{N}}$ are assumed to be given periodic sequences of integers. That is, for all i we have $r_{i+t} = r_i$ for some $t \in \mathbb{N}$ and $s_{i+t'} = s_i$ for some $t' \in \mathbb{N}$. These sequences will be fixed throughout this paper, and any dependence on them will oftentimes be implicit. One of the main results we will obtain is that, assuming Schanuel's conjecture, there exist, for every fixed a, infinitely many b such that $v_{a,b} < v_{a,b-1}$. Unfortunately, we don't get any explicit upper bound on the smallest b for which this holds, because we will use an ineffective version of Kronecker's Approximation Theorem in the process. So to compensate and as a warm-up, we will first prove an effective, unconditional version that $v_{a,b} < v_{a,b-1}$ holds infinitely often, in a special case (which is conjectured to be completely general though). This special case roughly asserts that there exists a large prime p that divides $u_{1,n}$, for some n. And, as we will see, it is immediate that this is true when $r_i = s_i = 1$ for all

i, and it is through this Theorem that we obtain the results stated in the abstract. Also, the proof of this unconditional special case is easier and more elementary than the general case, so this serves a good warm-up, especially because the high-level idea is the same: we are done if we have large prime dividing $u_{1,n}$. The techniques used will, from time to time, also be similar. After we are done with these upper bounds, we will look at lower bounds for the smallest b for which $v_{a,b} < v_{a,b-1}$. We will in particular prove that $v_{a,b}$ is a monotone increasing function in b, as long as $b < a + (\frac{1}{2} - \epsilon)\log a$. In our last section we will then prove that this lower bound can't be improved in general, apart from the constant in front of $\log a$. More precisely, when $r_i = s_i = 1$ for all i, then there exist infinitely many a such that the smallest b for which $v_{a,b} < v_{a,b-1}$, is smaller than $a + (2+\epsilon)\log a$.

2. Further Notation and Definitions

In this note, $L_{a,b}$ will denote the least common multiple of all integers in the set $\{as_a, (a+1)s_{a+1}, .., bs_b\}$ and we set $L_n = L_{1,n}$. Further, $X_{a,b}$ will be defined as $L_{a,b}\sum_{i=a}^{b}\frac{r_i}{is_i}$, and, similarly, $X_n = X_{1,n}$. Remember that $\{r_i\}_{i\in\mathbb{N}}$ and $\{s_i\}_{i\in\mathbb{N}}$ are given periodic sequences of integers with period t and t', respectively. We will, without loss of generality, assume $t = t'$ throughout and assume s_i is positive for all i. We set $s^* = \max_i s_i$ and $r^* = \max_i |r_i|$, where it is assumed that $r^* \geq 1$, to avoid the trivial case. As stated in the introduction, the dependence of, for example, $L_{a,b}$ and $X_{a,b}$ on r_i, s_i and t will usually be implicit. Furthermore, $\lambda = \lambda(t)$ is defined as the smallest positive integer such that, for all p coprime to t, we have $p^\lambda \equiv 1 \pmod t$ while, of course, Euler's Totient Theorem implies that λ exists. The letter p is reserved for prime numbers and every other letter used will, unless stated otherwise, always denote an integer (usually non-negative). Also, $O(f(x))$ and $o(f(x))$ are the familiar Big-O and Little-o notations, $x|y$ reads 'x divides y', $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N}$ for the set of positive integers. Last, but not least, $b(a)$ is the smallest integer $b > a$ such that $v_{a,b} < v_{a,b-1}$.

3. Upper Bounds

Theorem 1. *If a positive integer* $n = lp^k$ *and a prime* $p > \max(ls_n, r^*, s^*, t)$ *exist such that* p *divides* X_n *and* $r_n \neq 0$, *then infinitely many* b *exist with:* $v_{a,b} < v_{a,b-1}$. *Furthermore, we then have:* $b(a) = O(a)$. *More precisely,* $b(a) \leq lp^\lambda$ *if* $a = 1$ *and* $b(a) \leq lp^\lambda(a-1)$ *for all* $a > 1$. [1]

Proof. In general we have: $\dfrac{u_{a,b}}{v_{a,b}} = \dfrac{X_{a,b}}{L_{a,b}}$, so if we define $g_{a,b}$ to be the greatest common divisor of $X_{a,b}$ and $L_{a,b}$, then $v_{a,b} = \dfrac{L_{a,b}}{g_{a,b}}$. And thus, if $L_{a,b} = L_{a,b-1}$, then $v_{a,b} < v_{a,b-1}$ holds true, precisely when $g_{a,b} > g_{a,b-1}$.

Now we will introduce the most important Lemma of this paper, which will be used again and again. To be able to state it, let $c_0 = c_0(p)$ be such that p^{c_0} exactly divides t, let $c_1 = c_1(p)$ be the smallest non-negative integer for which, for all i, p^{c_1+1} doesn't divide s_i, and let $c = \max(c_0, c_1)$. We then have:

Lemma 1. *Let* k *and* m *be any non-negative integers and let* p *be any prime number. If* $p^{2c+d+m} \leq b$ *for some non-negative integer* d, *and if* a *is smaller than or equal to* p^{c+d} *and if* a' *is any integer smaller than or equal to* $p^{\lambda k+c+d}$ *and* b' *is any integer with* $bp^{\lambda k} \leq b' < (b+1)p^{\lambda k}$, *then:* $p^{m+1}|X_{a,b}$ *if, and only if,* $p^{m+1}|X_{a',b'}$. [2]

Proof. First we will prove that $p^{m+1}|X_{a,b}$ if, and only if, $p^{m+1}|X_{a',bp^{\lambda k}}$. Then we will show that if $bp^{\lambda k} < b' < (b+1)p^{\lambda k}$, then $p^{m+1}|X_{a',b'}$ and $p^{m+1}|X_{a',b'-1}$ are equivalent. The proof is mainly based on the observation that if $\dfrac{L_{a,b}r_i}{is_i}$ doesn't vanish modulo p^{m+1}, then, since p^{c+1} doesn't divide s_i, i must be divisible by the power of p that divides $L_{a,b}p^{-(c+m)}$. In particular, i must be divisible by p^{c+d}. This implies:

[1]Note that Theorem 1 already implies everything that was claimed in the abstract. In the special case $r_1 = s_1 = t = 1$, we may use $p = 3, n = l = 2$, as we then have: $3|X_2$. So, if $a > 1 = r_1 = s_1 = t$, then, indeed, $b(a) \leq 6(a-1)$. Furthermore, $v_{a,b} < v_{a,b-1}$ implies $v_{a,b} < L_{a,b}$, which can only happen when $\gcd(X_{a,b}, L_{a,b}) > 1$.

[2]Apart from the proof of Lemma 5, we only need the case $c = m = 0$. Also, we basically only care about $a' = a$, or $a' = 1$ and $b' = b$.

$$
\begin{aligned}
X_{a,b} &= L_{a,b} \sum_{i=a}^{b} \frac{r_i}{is_i} \\
&\equiv L_{a,b} \sum_{i=1}^{\lfloor bp^{-(c+d)} \rfloor} \frac{r_{ip^{c+d}}}{ip^{c+d} s_{ip^{c+d}}} \pmod{p^{m+1}} \\
&\equiv \frac{L_{a,b}}{p^{c+d}} \sum_{i=1}^{\lfloor bp^{-(c+d)} \rfloor} \frac{r_{ip^{c+d}}}{is_{ip^{c+d}}} \pmod{p^{m+1}}
\end{aligned}
$$

And similarly:

$$
\begin{aligned}
X_{a',bp^{\lambda k}} &= L_{a',bp^{\lambda k}} \sum_{i=a'}^{bp^{\lambda k}} \frac{r_i}{is_i} \\
&\equiv L_{a',bp^{\lambda k}} \sum_{i=1}^{\lfloor bp^{-(c+d)} \rfloor} \frac{r_{ip^{\lambda k+c+d}}}{ip^{\lambda k+c+d} s_{ip^{\lambda k+c+d}}} \pmod{p^{m+1}} \\
&\equiv \frac{L_{a',bp^{\lambda k}}}{p^{\lambda k+c+d}} \sum_{i=1}^{\lfloor bp^{-(c+d)} \rfloor} \frac{r_{ip^{c+d}}}{is_{ip^{c+d}}} \pmod{p^{m+1}}
\end{aligned}
$$

So to prove that $p^{m+1}|X_{a,b}$ if, and only if, $p^{m+1}|X_{a',bp^{\lambda k}}$, it suffices to show that the largest power of p that divides $\frac{L_{a,b}}{p^{c+d}}$ is equal to the largest power of p that divides $\frac{L_{a',bp^{\lambda k}}}{p^{\lambda k+c+d}}$. Or, equivalently, it suffices to show that the largest power of p that divides $L_{a,b}$ is equal to the largest power of p that divides $\frac{L_{a',bp^{\lambda k}}}{p^{\lambda k}}$.

To see that it can't be larger, let $j \leq b$ be such that the largest power of p that divides js_j equals the largest power of p that divides $L_{a,b}$. Note that, by the fact that $p^{2c} \leq b$, j must then be divisible by p^c, which implies that $s_{jp^{\lambda k}} = s_j$. And thus, since $jp^{\lambda k} \leq bp^{\lambda k}$, $L_{a',bp^{\lambda k}}$ must be divisible by $jp^{\lambda k} s_{jp^{\lambda k}} = jp^{\lambda k} s_j$.

To see that it can't be smaller, we use the same reasoning in reverse; let $j \leq bp^{\lambda k}$ be such that the largest power of p that divides js_j equals the largest power of p that divides $L_{a',bp^{\lambda k}}$. Note that j must now be divisible by $p^{\lambda k+c}$, which implies that $s_{jp^{-\lambda k}}$ exists and must equal s_j. And thus, $L_{a,b}$ must be divisible by $jp^{-\lambda k} s_{jp^{-\lambda k}} = jp^{-\lambda k} s_j$. And this concludes our proof of the fact that $p^{m+1}|X_{a,b}$ and $p^{m+1}|X_{a',bp^{\lambda k}}$ are equivalent.

Now, let b' be such that $bp^{\lambda k} < b' < (b+1)p^{\lambda k} = bp^{\lambda k} + p^{\lambda k}$. We will prove that $p^{m+1}|X_{a',b'}$ and $p^{m+1}|X_{a',b'-1}$ are equivalent, finishing our proof of Lemma 1.

To do this, note that, in general, $X_{a',b'} = \frac{X_{a',b'-1}L_{a',b'}}{L_{a',b'-1}} + \frac{L_{a',b'}r_{b'}}{b's'_b}$. First we will show that $\frac{L_{a',b'}r_{b'}}{b's'_b}$ vanishes modulo p^{m+1}, implying that we have: $X_{a',b'} \equiv \frac{X_{a',b'-1}L_{a',b'}}{L_{a',b'-1}} \pmod{p^{m+1}}$. Since $b' > bp^{\lambda k} \geq p^{\lambda k+2c+d+m} \geq a'$, $L_{a',b'}$ is divisible by $p^{\lambda k+2c+d+m}$. While on the other hand, b' is not divisible by $p^{\lambda k}$, which shows us that $b's_{b'}$ is not divisible by $p^{c+\lambda k}$, so $\frac{L_{a',b'}r_{b'}}{b's'_b}$ must be divisible by $p^{c+d+m+1}$ and, in particular, vanishes modulo p^{m+1}. So, indeed, $X_{a',b'} \equiv \frac{X_{a',b'-1}L_{a',b'}}{L_{a',b'-1}} \pmod{p^{m+1}}$. Furthermore, this suffices if p doesn't divide $\frac{L_{a',b'}}{L_{a',b'-1}}$, because then $X_{a',b'} \equiv 0 \pmod{p^{m+1}}$ if, and only if, $X_{a',b'-1} \equiv 0 \pmod{p^{m+1}}$. And indeed we will see that p doesn't divide $\frac{L_{a',b'}}{L_{a',b'-1}}$, by using the same reasoning. Because if $bp^{\lambda k} < b' < bp^{\lambda k} + p^{\lambda k}$, then, again, b' is not divisible by $p^{\lambda k}$, which implies that $b's_{b'}$ is, again, not divisible by $p^{c+\lambda k}$. While on the other hand, $L_{a',b'-1}$ is still divisible by $p^{\lambda k+2c+d+m}$, which is thus larger than the largest power of p that divides $b's_{b'}$. So the largest power of p that divides $L_{a',b'} = \operatorname{lcm}(L_{a',b'-1}, b's_{b'})$ is the same as the largest power of p that divides $L_{a',b'-1}$. □

With the help of Lemma 1 we can begin our construction of infinitely many b for which $v_{a,b} < v_{a,b-1}$. To that end, let $n = lp^{k'}$ be a positive integer such that a prime p larger than $\max(ls_n, r^*, s^*, t)$ exists that divides X_n and such that $r_n \neq 0$. Note that $l > 1$, because $X_{p^{k'}} = \sum_{i=1}^{p^{k'}} \frac{L_{p^{k'}} r_i}{is_i} \equiv \frac{L_{p^{k'}} r_{p^{k'}}}{p^{k'} s_{p^{k'}}} \pmod{p} \neq 0 \pmod{p}$. Set $b = np^{\lambda k} = lp^{\lambda k+k'}$, such that k is any integer for which $p^{\lambda k+k'} \geq \max(a, 2)$. Now we shall see that $v_{a,b} < v_{a,b-1}$, proving the first part of Theorem 1. For the second part, observe that for the smallest possible k, we have $p^{\lambda k+k'-\lambda} \leq \max(a-1, 1)$, and thus $b = lp^{\lambda k+k'} \leq \max(a-1,1)lp^{\lambda}$.

By Lemma 1, we have: $p|X_{a,b}$. We furthermore claim that $L_{a,b} = L_{a,b-1}$. This holds because $\gcd(p^{\lambda k+k'}, ls_b) = 1$, while $p^{\lambda k+k'} \in \{a, a+1, .., b\}$ and $ls_b = ls_n < p \leq b-a$, which implies ls_b is a divisor of some integer in the set $\{a, a+1, .., b\}$. And thus: $L_{a,b} = \operatorname{lcm}(bs_b, L_{a,b-1}) = \operatorname{lcm}(lp^{\lambda k+k'}s_b, L_{a,b-1}) = L_{a,b-1}$. So to prove Theorem 1, it suffices to show $g_{a,b} > g_{a,b-1}$. Now we need a Lemma.

Lemma 2. *Let a, b, c, d be integers such that c divides ab. Then:* $\gcd(a, \frac{ab}{c} + d)$ *is an integer multiple of* $\frac{\gcd(a,d)}{\gcd(c,d)}$.

Proof. To prove this, let p be any prime dividing a and let $p^\alpha, p^\beta, p^\gamma, p^\delta$ be the largest powers of p dividing a, b, c, d respectively (note: $\alpha + \beta \geq \gamma$). If $\min(\gamma, \delta) = \gamma$, the largest power of p that divides $\gcd(c, d)\gcd(a, \frac{ab}{c} + d)$ equals:

$$\begin{aligned}\gcd(p^{\alpha+\min(\gamma,\delta)}, p^{\alpha+\beta+\min(\gamma,\delta)-\gamma} + p^{\delta+\min(\gamma,\delta)}) &\geq \gcd(p^\alpha, p^{\alpha+\beta} + p^{\delta+\gamma})\\ &= \gcd(p^\alpha, p^{\delta+\gamma})\\ &\geq \gcd(p^\alpha, p^\delta)\end{aligned}$$

which equals the largest power of p that divides $\gcd(a, d)$, which we wanted to prove.

And if $\min(\gamma, \delta) = \delta$, the largest power of p that divides $\gcd(c, d)\gcd(a, \frac{ab}{c} + d)$ equals:

$$\begin{aligned}\gcd(p^{\alpha+\min(\gamma,\delta)}, p^{\alpha+\beta+\min(\gamma,\delta)-\gamma} + p^{\delta+\min(\gamma,\delta)}) &\geq \gcd(p^\alpha, p^{\alpha+\beta+\delta-\gamma} + p^{2\delta})\\ &\geq \gcd(p^\alpha, p^\delta + p^{2\delta})\\ &\geq \gcd(p^\alpha, p^\delta)\end{aligned}$$

which, again, equals the largest power of p that divides $\gcd(a, d)$. □

Now, we know that $p|X_{a,b}$, while, since $\frac{L_{a,b}r_b}{bs_b}$ doesn't vanish modulo p, p doesn't divide $X_{a,b-1}$. Combining this with our knowledge of Lemma 2, finishes up our proof for Theorem 1:

$$
\begin{aligned}
g_{a,b} &= \gcd(L_{a,b}, X_{a,b}) \\
&= \gcd(p^{\lambda k+k'}, X_{a,b}) * \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b}\right) \\
&\geq p \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b}\right) \\
&= p \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1} + \frac{L_{a,b} r_b}{b s_b}\right) \\
&= p \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1} + \frac{L_{a,b} r_b}{l p^{\lambda k+k'} s_b}\right) \\
&\geq \frac{p}{\gcd(X_{a,b-1}, l s_b)} \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&\geq \frac{p}{l s_b} \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&= \frac{p}{l s_n} \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&> \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&= \gcd\left(\frac{L_{a,b-1}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&= \gcd(L_{a,b-1}, X_{a,b-1}) \\
&= g_{a,b-1}
\end{aligned}
$$
□

Note that, by the same reasoning, we can actually show that $l s_b g_{a,b}$ is a multiple of $p g_{a,b-1}$. We end this section with a conjecture that, if true, would imply that Theorem 1 is completely general;

Conjecture 1. *For every $c \in \mathbb{R}$ there exists a positive integer $n = lp^k$ and a prime number p, such that: $p > cl$ and $p | X_n$.*[3]

[3]There is a follow-up paper planned in which we prove this for $c = 1$, which suffices for the case $s^* = 1$.

4. Upper Bounds under Schanuel's Conjecture

Since Conjecture 1 may be out of reach for the moment, we proceed along a different path, assuming (the need for this will be clear later) the following for some results in this section:

Conjecture 2. *(Schanuel's Conjecture) If $z_1, .., z_m$ are rationally independent reals, then at least m of the following are algebraically independent: $z_1, .., z_m, e^{z_1}, .., e^{z_m}$.*

Now, by the argument in the previous section, we still do know that if a large p divides X_b and doesn't divide X_{b-1} then, assuming a power of p lies between a and b, we have that $pg_{a,b-1}$ divides $ls_b g_{a,b}$. But this doesn't imply $g_{a,b} > g_{a,b-1}$, if we are not allowed to assume $p > ls_b$. So our strategy is as follows: we want a large prime p dividing $X_{a,b}$, while $\gcd(ls_b, X_{a,b-1})$ is smaller than some constant q, say, implying with the use of Lemma 2 that we actually have that $pg_{a,b-1} < qg_{a,b}$. And, of course, if p is larger than q, this suffices. So to carry out the aforementioned strategy, we thus need two things: we need a large prime p dividing X_b and we need $\gcd(ls_b, X_{a,b-1})$ to be small. Let's start with finding such a large prime p.

Theorem 2. *If Conjecture 2 holds, then there exist arbitrarily large primes p, such that $p|X_n$, for some $n = lp^k$, where $1 < l < p$ and $r_n \neq 0$.*

Before we start our proof, note that Theorem 1 and Theorem 2 together imply the following:

Theorem 3. *If Conjecture 2 holds and $s^* = 1$, then there exist infinitely many b such that $v_{a,b} < v_{a,b-1}$. Furthermore, we then have: $b(a) = O(a)$. More precisely, there exists a prime p and a positive integer $l < p$, such that $b(a) \leq lp^\lambda$ if $a = 1$ and $b(a) \leq lp^\lambda(a-1)$ for all $a > 1$.*

Proof of Theorem 2. To prove this, we first have to show that X_n itself gets large fast enough;

Lemma 3. $|X_n| = e^{(1+o(1))n}$.

Proof. By the prime number theorem, $L_n = e^{(1+o(1))n}$. An upper bound on $|X_n|$ is now easily established:

$$\begin{aligned}
|X_n| &= \left| L_n \sum_{i=1}^{n} \frac{r_i}{is_i} \right| \\
&\leq L_n \sum_{i=1}^{n} \left| \frac{r_i}{is_i} \right| \\
&\leq L_n \sum_{i=1}^{n} \frac{r^*}{i} \\
&\leq L_n r^* n \\
&= e^{(1+o(1))n}
\end{aligned}$$

So a lower bound equal to $e^{(1+o(1))n}$ suffices. This can be done by noting that $\sum_{i=n+1}^{n+t} \frac{r_i}{is_i}$ equals $\frac{f(n)}{g(n)}$, where $f(n)$ and $g(n)$ are polynomials with integral coefficients and degree at most t. If the leading coefficients of $f(n)$ and $g(n)$ have the same sign, then $\frac{f(n)}{g(n)}$ is positive for all large n, and if the leading coefficients of $f(n)$ and $g(n)$ differ in sign, then $\frac{f(n)}{g(n)}$ is negative for all large n. Either way, this implies that the sequence $\frac{X_n}{L_n}, \frac{X_{n+t}}{L_{n+t}}, \frac{X_{n+2t}}{L_{n+2t}}, ..$ is monotonic, for large enough n. If this sequence doesn't converge to zero, we are clearly done. If it does converge to zero, we have (for some absolute constant c and large enough n):

$$\begin{aligned}
\left| \frac{X_n}{L_n} \right| &= \left| \frac{X_n}{L_n} - 0 \right| \\
&> \left| \frac{X_n}{L_n} - \frac{X_{n+t}}{L_{n+t}} \right| \\
&= \left| \frac{f(n)}{g(n)} \right| \\
&> cn^{-t}
\end{aligned}$$

And since $cn^t = e^{o(n)}$, this proves Lemma 3. □

So for large enough n, we may suppose X_n is large. Now we will use this to show that the prime divisors of X_n must also be large sometimes. To accomplish this, remember that c_0 was such that p^{c_0} exactly divides t, and let us define c_4 to be the smallest positive integer for which $r_{c_4} \neq 0$. Also, let $\Sigma_1, \Sigma_2, \Sigma_3$ be defined as follows:

1. $\Sigma_1 = \{p : p > \max(r^*, s^*, t)\}$

2. $\Sigma_2 = \{p : p \le \max(r^*, s^*, t),$ and $r_{ip^{c_0}} \neq 0$ for some $i\}$

3. $\Sigma_3 = \{p : p \le \max(r^*, s^*, t),$ and $r_{ip^{c_0}} = 0$ for all $i\}$

We then have the following three important Lemmata:

Lemma 4. *If $p \in \Sigma_1$, then there exists a positive integer c_p, such that for all $k \in \mathbb{N}$ and all n with: $c_p p^{\lambda k} \le n < (c_p+1)p^{\lambda k}$, we have that p doesn't divide X_n.*

Lemma 5. *If $p \in \Sigma_2$, then there exists a positive integer c_p, such that for all $k \in \mathbb{N}$ and all n with: $c_p p^{\lambda k} \le n < (c_p+1)p^{\lambda k}$, we have that $p^{5r^*s^*t}$ doesn't divide X_n.*

Lemma 6. *If $p \in \Sigma_3$, then for all n with: $n \equiv c_4 \pmod{t^{7r^*s^*t}}$, we have that $p^{3r^*s^*t+\lfloor \frac{\log n}{\log p} \rfloor}$ doesn't divide X_n.*

The proofs of these three Lemmata are somewhat technical, and could be skipped at first reading. The important things to grasp are the actual statements of the Lemmata and the fact that they, together, imply Lemma 7.

Proof of Lemma 4. Assume $p \in \Sigma_1$, set $c_p = c_4$ and note that $c_4 \le t$. So we then have that $X_{c_p} = \dfrac{L_{c_p} r_{c_p}}{c_p s_{c_p}}$, which has all its prime divisors smaller than or equal to $\max(r^*, s^*, t)$ and so, by assumption, p doesn't divide X_{c_p}. By Lemma 1 we have as a consequence that p doesn't divide X_n for all k and all n with: $c_p p^{\lambda k} \le n < (c_p+1)p^{\lambda k}$, which proves Lemma 4. □

Proof of Lemma 5. Assume $p \in \Sigma_2$, let i' be such that $r_{i'p^{c_0}} \neq 0$ and assume j is the smallest positive integer for which $j \equiv i' \left(p^{13r^*s^*t-c_0}\right)^{-1} \pmod{tp^{-c_0}}$. Now, set d to be j or $j + tp^{-c_0}$, which ever one is coprime to p, and set $n' = (d+ps^*t)p^{13r^*s^*t}$. Observe that d is smaller than or equal to $2t$, and $r_{n'} = r_{dp^{13r^*s^*t}} = r_{i'p^{c_0}} \neq 0$. Also, we claim that $L_{n'} = L_{n'-1}$. To see this, note that $d + ps^*t$ is not divisible by p. So it suffices to show that both $(d+ps^*t)s_{n'}$ and $p^{13r^*s^*t}s_{n'}$ divide $L_{n'-1}$. But both of these are smaller than n', so this is immediate. Also, $n' = (d+ps^*t)p^{13r^*s^*t} \le 2ps^*tp^{13r^*s^*t} < p^{3s^*t}p^{13r^*s^*t} \le p^{16r^*s^*t}$. So, since for all i, p^{s^*} doesn't divide s_i, the largest power of p that divides $L_{n'}$ is smaller than $p^{16r^*s^*t}p^{s^*} \le p^{17r^*s^*t}$, which in turn implies that the largest power of p that divides $L_{n'}r_{n'}$ is smaller than $p^{17r^*s^*t}p^{r^*} \le p^{18r^*s^*t}$. And since n' is divisible by $p^{13r^*s^*t}$, we have:

$$X_{n'} = X_{n'-1} + \frac{L_{n'}r_{n'}}{n's_{n'}} \not\equiv X_{n'-1} \pmod{p^{5r^*s^*t}}$$

So at least one of $X_{n'-1}$ and $X_{n'}$ is not divisible by $p^{5r^*s^*t}$. If $X_{n'}$ is not divisible by $p^{5r^*s^*t}$, we can invoke Lemma 1, and thereby proving Lemma 5 with $c_p = n' = (d+ps^*t)p^{13r^*s^*t}$. If $X_{n'-1}$ is not divisible by $p^{5r^*s^*t}$, we can also invoke Lemma 1, proving Lemma 5 with $c_p = n' - 1 = (d+ps^*t)p^{13r^*s^*t} - 1$. □

Proof of Lemma 6. Assume $p \in \Sigma_3$. First of all, if $n = c_4$, then $X_n = \dfrac{L_{c_4} r_{c_4}}{c_4 s_{c_4}}$. And since $c_4 \leq t$, the largest power of p that divides L_n is then smaller than p^{2s^*t}. So X_n is then not divisible by $p^{3r^*s^*t}$. So we may safely assume $n > c_4$. Secondly, since $r_{ip^{c_0}} = 0$ for all i, p must divide t. Because otherwise, c_0 would, by assumption, equal 0, which implies $r_i = 0$ for all i. So if we let $n \equiv c_4 \pmod{t^{7r^*s^*t}}$, then there definitely exists a positive integer c_6, such that $n = c_4 + c_6 t p^{6r^*s^*t}$. Let k be such that $p^{k+c_0+s^*}$ exactly divides L_n and note that, since the power of p that divides L_n is smaller than $p^{s^*+\lfloor \frac{\log n}{\log p} \rfloor}$, k must be smaller than $\left\lfloor \dfrac{\log n}{\log p} \right\rfloor$. Also, define L'_n to be $\dfrac{L_n}{p^k}$, so that $\dfrac{L'_n r_i}{i s_i}$ is still an integer for all $i \leq n$. We then claim that: $\dfrac{L'_n r_i}{i s_i} \equiv \dfrac{L'_n r_{i+tp^{k'}}}{(i+tp^{k'}) s_{i+tp^{k'}}} \pmod{p^{k'}}$ for all $i \leq n - tp^{k'}$ and all $k' \geq 0$. Of course, $r_i = r_{i+tp^{k'}}$ (which in particular shows the truth of our claim in the case $r_i = 0$) and $s_i = s_{i+tp^{k'}}$. So all we have to prove is $\dfrac{L'_n}{i} \equiv \dfrac{L'_n}{i+tp^{k'}} \pmod{p^{k'}}$ holds, whenever $r_i \neq 0$. This could only be false if the power of p that divides i is different from the power of p that divides $i + tp^{k'}$. And this can only happen when $p^{c_0+k'}$ divides i. But, by our assumption that $p \in \Sigma_3$, if $p^{c_0} | i$, then $r_i = 0$, and this proves our claim. This implies that we can split up $\displaystyle\sum_{i=1}^{n} \frac{L'_n r_i}{i s_i}$ in parts that are all equal modulo $p^{k'}$ for some suitable k'. And this yields:

$$\begin{aligned}
\frac{X_n}{p^k} &= \sum_{i=1}^{n} \frac{L'_n r_i}{i s_i} \\
&\equiv \frac{L'_n r_{c_4}}{c_4 s_{c_4}} + c_6 p^{3r^*s^*t} \left(\sum_{i=1}^{tp^{3r^*s^*t}} \frac{L'_n r_i}{i s_i} \right) && \pmod{p^{3r^*s^*t}} \\
&\equiv \frac{L'_n r_{c_4}}{c_4 s_{c_4}} && \pmod{p^{3r^*s^*t}} \\
&\neq 0 && \pmod{p^{3r^*s^*t}}
\end{aligned}$$

□

Let $q_1 = (3r^*s^*t)^{18(r^*s^*t)^2}$ and let q_2 be any real number. An immediate corollary to the previous three Lemmas, is the following:

Lemma 7. *If $n \equiv c_4 \pmod{t^{7r^*s^*t}}$, and if for all $p \leq q_2$ with $p \in \Sigma_1 \cup \Sigma_2$ there exists a $k = k(p)$, such that $c_p p^{\lambda k} < n < (c_p + 1) p^{\lambda k}$, then the largest divisor of X_n that has all its prime divisors smaller than or equal to q_2, is smaller than $q_1 n^t$. In particular, when $X_n > q_1 n^t$, then X_n has a prime divisor larger than q_2.*

Proof. Assume that all conditions in Lemma 7 are fulfilled and let $\Sigma_1^{q_2}$ denote the set of all primes that belong to Σ_1 and that are smaller than or equal to q_2. By Lemma 4, X_n is not divisible by primes from $\Sigma_1^{q_2}$. Further, we trivially have $|\Sigma_2| \leq \max(r^*, s^*, t) \leq 3r^*s^*t$ and we know by Lemma 5 that $p^{5r^*s^*t}$ doesn't divide X_n if $p \in \Sigma_2$. So if $d'|X_n$ and d' has all its prime divisors in $\Sigma_1^{q_2} \cup \Sigma_2$, then d' is smaller than:

$$\begin{aligned}\prod_{p\in\Sigma_2} p^{5r^*s^*t} &< \prod_{p\in\Sigma_2} (3r^*s^*t)^{5r^*s^*t} \\ &= \left((3r^*s^*t)^{5r^*s^*t}\right)^{|\Sigma_2|} \\ &\leq \left((3r^*s^*t)^{5r^*s^*t}\right)^{3r^*s^*t} \\ &= (3r^*s^*t)^{15(r^*s^*t)^2}\end{aligned}$$

[4]

Also, since $p|t$ for all $p \in \Sigma_3$, we have $|\Sigma_3| < \prod_{p\in\Sigma_3} p \leq t$. And we know by Lemma 6 that $p^{3r^*s^*t+\lfloor \frac{\log n}{\log p} \rfloor}$ doesn't divide X_n if $p \in \Sigma_3$. So if $d|X_n$ and d has all its prime divisors in $\Sigma_1^{q_2} \cup \Sigma_2 \cup \Sigma_3$, then d is smaller than:

$$\begin{aligned}(3r^*s^*t)^{15(r^*s^*t)^2} \prod_{p\in\Sigma_3} p^{3r^*s^*t+\lfloor \frac{\log n}{\log p} \rfloor} &\leq (3r^*s^*t)^{15(r^*s^*t)^2} \prod_{p\in\Sigma_3} p^{3r^*s^*t} \prod_{p\in\Sigma_3} n \\ &\leq (3r^*s^*t)^{15(r^*s^*t)^2} * t^{3r^*s^*t} * n^{|\Sigma_3|} \\ &< (3r^*s^*t)^{18(r^*s^*t)^2} * n^t\end{aligned}$$

□

Of course, by Lemma 3, for all large n the condition $X_n > q_1 n^t$ is satisfied. We shall prove that, independent of q_2, the two conditions of Lemma 7 can infinitely often be simultaneously satisfied, with the help of Schanuel's Conjecture and Kronecker's Approximation Theorem. And it is clear that this suffices for proving the first half of the statement of Theorem 2. Let $2 = p_1 < p_2 < .. < p_m$ be the sequence of primes up to q_2, and let c_{p_j} be the constant from Lemma 4 or Lemma 5 corresponding to the prime p_j. Now, if for some arbitrarily large $k = k(p_j)$ and n and all j with: $1 \leq j \leq m$, the following holds, we are done:

[4]Note that the argument up till now goes through for X_{n-1}, just as well as it did for X_n, because we assumed $n > c_p p^{\lambda k}$, which of course implies $n - 1 \geq c_p p^{\lambda k}$, which by Lemma 4 and 5 suffices. We will use this fact later on.

$$c_{p_j} p_j^{\lambda k} < n < (c_{p_j} + 1) p_j^{\lambda k} - (t^{7r^* s^* t}) \tag{1}$$

And indeed, we will see that there exist such k and n. Even more, if p is any prime larger than q_2 and l is any positive integer, then there exists infinitely many k', such that $n = lp^{\lambda k'}$ solves all m equations defined by (1). To prove this, we will define the following constants:

$$\alpha_j = \frac{\log p}{\log p_j}$$

$$\beta_j = \frac{2\log l - \log c_{p_j} - \log\left(c_{p_j} + 1 - \frac{t^{7r^* s^* t}}{p_j^{t^{7r^* s^* t}}}\right)}{2\lambda \log p_j}$$

$$\epsilon_j = \frac{\log\left(c_{p_j} + 1 - \frac{t^{7r^* s^* t}}{p_j^{t^{7r^* s^* t}}}\right) - \log c_{p_j}}{2\lambda \log p_j}$$

These are only defined to make sure we end up with manageable expressions. The important thing to note is that $\epsilon_j > 0$, for all j. Now, if we can satisfy the following equations infinitely often, we can solve all m equations defined by (1), for some n of the form $n = lp^{\lambda k'}$;

$$c_{p_j} p_j^{\lambda k} < lp^{\lambda k'} < (c_{p_j} + 1) p_j^{\lambda k} - (t^{7r^* s^* t})$$

$$c_{p_j} p_j^{\lambda k} < lp^{\lambda k'} < p_j^{\lambda k}\left(c_{p_j} + 1 - \frac{t^{7r^* s^* t}}{p_j^{\lambda k}}\right)$$

$$c_{p_j} p_j^{\lambda k} < lp^{\lambda k'} < p_j^{\lambda k}\left(c_{p_j} + 1 - \frac{t^{7r^* s^* t}}{p_j^{t^{7r^* s^* t}}}\right)$$

$$\lambda k \log p_j + \log c_{p_j} < \lambda k' \log p + \log l < \lambda k \log p_j + \log\left(c_{p_j} + 1 - \frac{t^{7r^* s^* t}}{p_j^{t^{7r^* s^* t}}}\right)$$

$$\lambda k \log p_j < \lambda k' \log p + \log l - \log c_{p_j} < \lambda k \log p_j + \log\left(c_{p_j} + 1 - \frac{t^{7r^* s^* t}}{p_j^{t^{7r^* s^* t}}}\right) - \log c_{p_j}$$

$$k < \left(\frac{\log p}{\log p_j}\right) k' + \beta_j + \epsilon_j < k + 2\epsilon_j$$

$$||\alpha_j k' + \beta_j|| < \epsilon_j \tag{2}$$

where $||x||$ denotes the distance from x to the nearest integer. By Kronecker's Approximation Theorem, there exist infinitely many k' such that the m equations defined by (2) are simultaneously satisfied, if the α_j together with 1 are rationally independent. And this is where Conjecture 2 comes in:

Lemma 8. *Assume Conjecture 2 holds. Then the* m *terms* $\dfrac{\log p}{\log p_j}$ *are, together with* 1*, rationally independent.*

Proof. [5] Since by unique factorization we have that all the terms $\log p_i$, are rationally independent, Conjecture 2 shows that they must in fact be algebraically independent, which immediately gives the result. □

So all the conditions from Lemma 7 can, independent from q_2, infinitely often be simultaneously satisfied, which proves the first half of the statement of Theorem 2.

Now that we have shown that X_n has arbitrarily large prime divisors, we will finish up our proof of Theorem 2, by proving the following Lemma:

Lemma 9. *If* $p > \max(r^*, s^*)$ *and* $n = lp^k$ *(where* $\gcd(l, p) = 1$*) is the smallest positive integer for which* p *divides* X_n *and* $X_{n-1} \neq 0 \neq X_n$*, then* $1 < l < p$ *and* $r_n \neq 0$.

Proof. Clearly, if $n = lp^k$ is the smallest positive integer for which p divides $X_n \neq 0$, then we immediately have $r_n \neq 0$. Now we assume $p > \max(r^*, s^*)$. In general we have: $X_n = \dfrac{X_{n-1}L_n}{L_{n-1}} + \dfrac{L_n r_n}{n s_n}$. If $l = 1$, n is a perfect power of p, and p therefore divides $\dfrac{L_n}{L_{n-1}}$. In particular we have: $X_n \equiv \dfrac{L_n r_n}{n s_n} \pmod p$. But since we in this case have $n = p^k$, the power of p that divides L_n is equal to the power of p that divides n. And this implies $X_n \equiv \dfrac{L_n r_n}{n s_n} \neq 0 \pmod p$, contrary to the assumption that p divides X_n. So $l = 1$ is impossible. Conversely, this implies that p does not divide $\dfrac{L_n}{L_{n-1}}$. So if $\dfrac{L_n r_n}{n s_n}$ were to vanish modulo p, and we know that $X_{n-1} \neq 0 \pmod p$ by definition of n, we have $X_n = \dfrac{X_{n-1}L_n}{L_{n-1}} \neq 0 \pmod p$, which is, again, contrary to the assumption that p divides X_n. And thus, $\dfrac{L_n r_n}{n s_n}$ may not vanish modulo p. So the largest power of p that divides L_n, must also divide n, which shows us that $n = lp^k < p^{k+1}$, or, equivalently, $l < p$. □

[5] this proof was shown to the author on www.mathoverflow.net, by Qiaochu Yuan and Matt Papanikolas

Now we are ready to prove, although, because of the ineffectiveness of Kroneckers Approximation Theorem, without an explicit upper bound, the finiteness of $b(a)$ in full generality:

Theorem 4. *If Conjecture 2 holds, then there exist infinitely many* b *such that* $v_{a,b} < v_{a,b-1}$.

Proof. To prove this, we use Theorem 2 to get the smallest n such that $n = lp^{k'}$ with $p|X_n$, and where p (which until the end of this proof is now fixed) is any prime larger than q_1. Since n is the smallest positive integer for which $p|X_n$, we have that $r_n \neq 0$. In particular, n, and thus l, is not divisible by $p_i^{c_0}$, where p_i is any prime inside Σ_3. That is, the largest divisor of l that is composed of primes contained in Σ_3 is smaller than $\prod_{p_i \in \Sigma_3} p_i^{c_0} \leq t$. Also, we set $q_2 = l$ in Lemma 7 and use our proof that all conditions of Lemma 7 can now be simultaneously satisfied for some large $b = np^{\lambda k} = lp^{\lambda k + k'}$. This implies that the largest divisor of X_{b-1} that is composed of primes contained in $\Sigma_1^l \cup \Sigma_2$ is smaller than $(3r^*s^*t)^{15(r^*s^*t)^2}$ [6]. And if b is large enough, this implies, by Lemma 1, that $\gcd(X_{a,b-1}, ls_b) \leq s^* \gcd(X_{a,b-1}, l) < s^*t(3r^*s^*t)^{15(r^*s^*t)^2} < q_1 < p$, and thus:

[6]Here we use the fact that the first part of the proof of Lemma 7 goes through for both X_n and X_{n-1}.

$$
\begin{aligned}
g_{a,b} &= \gcd(L_{a,b}, X_{a,b}) \\
&= \gcd(p^{\lambda k+k'}, X_{a,b}) * \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b}\right) \\
&\geq p \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b}\right) \\
&= p \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1} + \frac{L_{a,b} r_b}{b s_b}\right) \\
&= p \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1} + \frac{L_{a,b} r_b}{l p^{\lambda k+k'} s_b}\right) \\
&\geq \frac{p}{\gcd(X_{a,b-1}, l s_b)} \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&\geq \frac{p}{q_1} \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&> \gcd\left(\frac{L_{a,b}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&= \gcd\left(\frac{L_{a,b-1}}{p^{\lambda k+k'}}, X_{a,b-1}\right) \\
&= \gcd(L_{a,b-1}, X_{a,b-1}) \\
&= g_{a,b-1}
\end{aligned}
$$

□

5. A Uniform Lower Bound

Theorem 5. $\liminf_{a\to\infty} \frac{b(a)-a}{\log a} \geq \frac{1}{2}$

Proof. Let ϵ be a given small positive constant, assume that a is large enough in terms of ϵ and s^*, and let b be any integer such that $a < b < a + (\frac{1}{2} - \epsilon)\log a$. Then we shall see that $v_{a,b} > v_{a,b-1}$. To achieve this, recall that $v_{a,b} = \frac{L_{a,b}}{\gcd(X_{a,b}, L_{a,b})}$. So $v_{a,b} > v_{a,b-1}$ precisely when $\frac{L_{a,b}}{L_{a,b-1}} > \frac{\gcd(X_{a,b}, L_{a,b})}{\gcd(X_{a,b-1}, L_{a,b-1})}$. We will show that this is indeed the case, by proving the following two inequalities: $\frac{L_{a,b}}{L_{a,b-1}} > \sqrt{b}$ and $\frac{\gcd(X_{a,b}, L_{a,b})}{\gcd(X_{a,b-1}, L_{a,b-1})} < \sqrt{b}$. This first inequality is established as follows:

$$\begin{aligned}
\frac{L_{a,b-1}}{L_{a,b}} &= \frac{L_{a,b-1}}{\operatorname{lcm}(L_{a,b-1}, bs_b)} \\
&\leq \frac{L_{a,b-1}}{\operatorname{lcm}(L_{a,b-1}, b)} \\
&= \frac{L_{a,b-1}\gcd(L_{a,b-1}, b)}{bL_{a,b-1}} \\
&= b^{-1}\gcd(L_{a,b-1}, b) \\
&\leq b^{-1}(s^*)! \prod_{p^k \leq b-a < p^{k+1}} \gcd(p^k, b) \\
&\leq b^{-1}(s^*)! \prod_{p^k \leq (\frac{1}{2}-\epsilon)\log a < p^{k+1}} p^k \\
&= b^{-1}(s^*)! e^{(\frac{1}{2}-\epsilon+o(1))\log a} \qquad (3) \\
&= b^{-1}\sqrt{a}\frac{(s^*)!}{a^{\epsilon-o(1)}} \\
&< b^{-1}\sqrt{a} \\
&< b^{-1}\sqrt{b} \\
&= \frac{1}{\sqrt{b}}
\end{aligned}$$

Where (3) is obtained as a consequence of the prime number theorem. For the second inequality, let p be a prime dividing $L_{a,b}$ and let $k = k(p)$ be such that p^k exactly divides $L_{a,b-1}$. Then p can belong to three different sets:

1. $\Sigma_1 = \{p : p \text{ is not a divisor of } bs_b\}$
2. $\Sigma_2 = \{p : p^{k+1} \text{ divides } bs_b\}$

3. $\Sigma_3 = \{p : p^{k'} \text{ divides } bs_b, \text{ for some } k' \text{ with } 1 \le k' \le k\}$

If $p \in \Sigma_1$, we note that $\dfrac{L_{a,b}}{L_{a,b-1}} \not\equiv 0 \pmod{p}$, while $\dfrac{L_{a,b}}{bs_b} \equiv 0 \pmod{p^k}$. We thus have:

$$\begin{aligned} X_{a,b} &= \frac{L_{a,b}X_{a,b-1}}{L_{a,b-1}} + \frac{L_{a,b}}{bs_b} \\ &\equiv \frac{L_{a,b}X_{a,b-1}}{L_{a,b-1}} \pmod{p^k} \end{aligned}$$

And, since $\dfrac{L_{a,b}}{L_{a,b-1}} \not\equiv 0 \pmod{p}$, the largest power of p that divides $\gcd(X_{a,b}, L_{a,b})$ is the same as the largest power of p that divides $\gcd(X_{a,b-1}, L_{a,b-1})$. So if $p \in \Sigma_1$, then p can be ignored in determining the size of $\dfrac{\gcd(X_{a,b}, L_{a,b})}{\gcd(X_{a,b-1}, L_{a,b-1})}$.

If $p \in \Sigma_2$ we have: $\dfrac{L_{a,b}}{L_{a,b-1}} \equiv 0 \not\equiv \dfrac{L_{a,b}}{bs_b} \pmod{p}$, and so:

$$\begin{aligned} X_{a,b} &= \frac{L_{a,b}X_{a,b-1}}{L_{a,b-1}} + \frac{L_{a,b}}{bs_b} \\ &\not\equiv 0 \pmod{p} \end{aligned}$$

Thus, trivially, p doesn't divide $\gcd(X_{a,b}, L_{a,b})$ either. This, also trivially, implies that the power of p that divides $\gcd(X_{a,b-1}, L_{a,b-1})$ must be at least as large as the power of p that divides $\gcd(X_{a,b}, L_{a,b})$. So primes inside Σ_2 can't be responsible either for making $\gcd(X_{a,b}, L_{a,b})$ larger than $\gcd(X_{a,b-1}, L_{a,b-1})$. They could only work *for* us, so to speak.

In conclusion, the only primes that can make $\gcd(X_{a,b}, L_{a,b})$ larger than $\gcd(X_{a,b-1}, L_{a,b-1})$ are the primes from Σ_3. So $\gcd(X_{a,b}, L_{a,b})$ can never be larger than $d\gcd(X_{a,b-1}, L_{a,b-1})$, where d is the largest divisor of bs_b, such that $p|d$ implies $p \in \Sigma_3$. So d must both be a divisor of bs_b and of $L_{a,b-1}$, and must thus be smaller than or equal to $s_b \gcd(L_{a,b-1}, b)$. And this is, by the same reasoning we used before, smaller than $\sqrt{b}$ [7], proving that we indeed have $\dfrac{\gcd(X_{a,b}, L_{a,b})}{\gcd(X_{a,b-1}, L_{a,b-1})} < \sqrt{b}$. □

[7]The only difference being that we now need that $\frac{s_b(s^*)!}{a^{\epsilon - o(1)}} < 1$, instead of $\frac{(s^*)!}{a^{\epsilon - o(1)}} < 1$, and this is, of course, again handled by just assuming that a is large enough.

6. Tightness of Lower Bound, up to a constant

Theorem 6. *If* $r_i = s_i = t = 1$*, then* $\liminf_{a\to\infty} \frac{b(a)-a}{\log a} \leq 2$

Proof. Let q' be the product of all primes between $\frac{3^k}{2}$ en 3^k and define $q = q'$ if $q' \equiv 2 \pmod 3$ and $q = 2q'$ if $q' \equiv 1 \pmod 3$. Note that, by the prime number theorem, $q = e^{(1/2+o(1))3^k}$. Now, choose $a = (q-1)3^k$ and set $b = a + 3^k$. We claim that $v_{a,b} < v_{a,b-1}$, which would imply that we have: $b(a) \leq b = a + 3^k = a + (2 + o(1)) \log a$. To prove that $v_{a,b} < v_{a,b-1}$, first observe that every prime power divisor of b is smaller than or equal to 3^k. And since $b - a = 3^k$, every prime power divisor of b is also a prime power divisor of some number between a and $b-1$ (inclusive). This implies that $L_{a,b} = L_{a,b-1}$. So to prove $v_{a,b} < v_{a,b-1}$, it suffices to show that $\gcd(X_{a,b}, L_{a,b}) > \gcd(X_{a,b-1}, L_{a,b-1})$. We now have to consider four distinct sets of primes:

1. $\Sigma_1 = \{p : p \text{ doesn't divide } b \text{ and } p > 3\}$.
2. $\Sigma_2 = \{p : p \text{ divides } b \text{ and } p > 3\}$.
3. $\{2\}$
4. $\{3\}$

Just like in our proof of the lower bound in the previous section, if $p \in \Sigma_1$, then p does not have any influence on the relative sizes of $\gcd(X_{a,b}, L_{a,b})$ and $\gcd(X_{a,b-1}, L_{a,b-1})$.

If $p \in \Sigma_2$, we have that $\frac{L_{a,b-1}}{i}$ vanishes modulo p, unless p divides i. But if $a \leq i \leq b-1$, then p divides i if, and only if, $i = b - p$, because $b - 2p < b - 3^k = a$. So we get:

$$\begin{aligned} X_{a,b-1} &= L_{a,b-1} \sum_{i=a}^{b} \frac{1}{i} \\ &\equiv \frac{L_{a,b-1}}{b-p} && \pmod p \\ &\neq 0 && \pmod p \end{aligned}$$

And we see that if $p \in \Sigma_2$, then p also can't be responsible for making $\gcd(X_{a,b}, L_{a,b})$ smaller than $\gcd(X_{a,b-1}, L_{a,b-1})$. These are again the primes that could only work *for* us.

For the prime 2, we use the well-known fact that among consecutive integers, there is exactly one that is divisible by a power of 2, such that no other number from that interval is also divisible by that power. In other words, $\frac{L_{a,b-1}}{i}$ vanishes modulo 2, for all but one value of i. This implies that $X_{a,n}$ is odd for all $n > a$, in particular when $n = b - 1$.

All there is left is the prime 3. By the same method we used before, we see that 3 doesn't divide $X_{a,b-1}$;

$$\begin{aligned} X_{a,b-1} &= L_{a,b-1} \sum_{i=a}^{b} \frac{1}{i} \\ &\equiv \frac{L_{a,b-1}}{a} && \pmod 3 \\ &\neq 0 && \pmod 3 \end{aligned}$$

But 3 does divide $X_{a,b}$, thereby concluding our proof:

$$\begin{aligned} X_{a,b} &= L_{a,b} \sum_{i=a}^{b} \frac{1}{i} \\ &\equiv \frac{L_{a,b}}{a} + \frac{L_{a,b}}{b} && \pmod 3 \\ &\equiv (2q-1)\frac{L_{a,b}}{q(q-1)3^k} && \pmod 3 \\ &\equiv 0 && \pmod 3 \end{aligned}$$

□

7. Remarks

It is easy to show that we can improve Theorem 4 to the slightly stronger: If Conjecture 2 holds, then $\liminf_{b\to\infty} \frac{v_{a,b}}{v_{a,b-1}} = 0$. All we have to do is use Theorem 2 to get a prime p such that $p|X_n$ and $p > \epsilon^{-1}q_1$. We then have $v_{a,b} < \epsilon v_{a,b-1}$ infinitely often. In fact, in the case $r_i = s_i = t = 1$, we can prove $\liminf_{b\to\infty} \frac{v_{a,b}}{v_{a,b-1}} = 0$ unconditionally and effectively. To this end, let p be the smallest prime which is 1 (mod 2^m). This exists by Dirichlet's Theorem on primes in arithmetic progressions, and we can even get an effective upper bound on p, by (an effective version of) Linnik's Theorem. Using Wolstenholme's Theorem and the same methods we used to prove Theorem 1, we can then show that if $b = (p-1)p^k > p^k \geq a$, then $v_{a,b} < 2^{-m}v_{a,b-1}$. For completeness' sake, the Theorem would then be:

Theorem 7. *If $r_i = s_i = 1$ for all i, then there exist effectively computable constants c_1, c_2 such that for every a and every $\epsilon \in (0,1]$, there exists $b \leq c_1\epsilon^{-c_2}a$ such that $v_{a,b} < \epsilon v_{a,b-1}$.*

On another note, it would be very nice to prove Conjecture 1 (not to mention Conjecture 2!) so that, in general, we get a linear upper bound $b(a) \leq ca$ for some constant c. There is a follow-up paper planned in which we will prove Conjecture 1 for the case $c = 1$, which leads to proofs of Theorem 3 and Theorem 4 *without* the assumption of Conjecture 2. Furthermore, in the case $s^* = 1$ we then find an explicit function for the constant in '$b(a) = O(a)$'. Also, a different question, which has been ignored in this article, is: how fast can f grow, such that $\limsup_{a\to\infty} \frac{b(a)-a}{f(a)} > 0$? It seems hard to even guess the truth here (can we show e.g. $f(a) > \log^2 a$?), but it is very likely that $f(a) = o(a)$. Or, in other words, $b(a) = O(a)$ is definitely not the whole truth. But, when $r_1 = s_1 = t = 1$, even to show that $b(a) < 6(a-1)$ for all $a \geq 5$ may need completely different ideas. On another note, it can be conjectured that $b(a-1) > b(a)$ happens infinitely often, which, at least when $r_1 = s_1 = t = 1$, shouldn't be too hard. Other questions also remain in the classical special case of $r_1 = s_1 = t = 1$. For example, it is still open if $\gcd(X_n, L_n) = 1$ holds for infinitely many n or not. This all boils down to how many l with $1 \leq l \leq p-1$ exist for which $\sum_{i=1}^{l} i^{-1} \equiv 0 \pmod{p}$. A decent upper bound on that number ought to resolve this question.

At last, let, for simplicity, $s_i = 1$ for all i. Ernie Croot then asked: what happens when we just assume that there exists a finite set A, consisting of non-zero integers, such that $r_i \in A$, for all i? As it turns out, then it *is* possible that the denominator of the corresponding sum grows monotonically. More precisely, we shall sketch a proof by induction that if $r_i \in \{-1, 1\}$, then it is possible that the denominator of $\sum_{i=1}^{n} \frac{r_i}{i}$ is never smaller than (and thus always equals) L_n. Recall that $X_n = L_n \sum_{i=1}^{n} \frac{r_i}{i}$ and note that what we need to prove is equivalent to the assertion that $X_n \not\equiv 0 \pmod{p}$ for all primes p with $p \leq n$. For $n = 1$ this is trivial, so assume that for some $n \geq 2$, we have, for all $p \leq n-1$, that $X_{n-1} \not\equiv 0 \pmod{p}$. We now have to distinguish between three different cases:

If $n = lp^k$ implies $l > p$, then it doesn't matter whether $r_n = 1$ or $r_n = -1$. Because we then have: $X_n = X_{n-1} \pm \frac{L_n}{n}$, and $\frac{L_n}{n}$ vanishes modulo p, for all $p < n$.

If $n = lp^k$ for some l with $1 < l < p$, then $\frac{L_n}{n}$ still vanishes modulo p' for all $p' < n$ different from p (this claim admits an easy proof), but it doesn't vanish modulo p. But, $\frac{L_n}{n} \not\equiv \frac{-L_n}{n} \pmod{p}$, since $p \neq 2$ (otherwise $l = 1$, contrary to our assumption). So we have that $X_{n-1} + \frac{L_n}{n} \not\equiv X_{n-1} - \frac{L_n}{n} \pmod{p}$. So either $X_{n-1} + \frac{L_n}{n} \not\equiv 0 \pmod{p}$, in which case we choose $r_n = 1$, or $X_{n-1} - \frac{L_n}{n} \not\equiv 0 \pmod{p}$, in which case we choose $r_n = -1$.

If $n = p^k$, then $X_n = pX_{n-1} \pm \frac{L_n}{n} \equiv pX_{n-1} \pmod{p'} \not\equiv 0 \pmod{p'}$, for all $p' < n$ different from p. And $X_n = pX_{n-1} \pm \frac{L_n}{n} \equiv \pm\frac{L_n}{n} \not\equiv 0 \pmod{p}$. So in this case it also doesn't matter whether $r_n = 1$ or $r_n = -1$.

This method can applied more generally, for any set of integers A. The most complete Theorem in that direction would then be:

Theorem 8. *If A is a set of integers containing at least one odd integer, and, for every odd prime p, there exist $a_1, a_2 \in A$ such that $a_1 \not\equiv a_2 \pmod{p}$, then it is possible to assign the r_i values in A, such that the denominator of $\sum_{i=1}^{n} \frac{r_i}{i}$ equals L_n for all $n \in \mathbb{N}$. Conversely, if for some set of integers A we have $r_i \in A$ for all i, and we know that the denominator of $\sum_{i=1}^{n} \frac{r_i}{i}$ is only finitely often smaller than L_n, then A must contain at least one odd integer, and, for every odd prime p, there exist $a_1, a_2 \in A$, such that $a_1 \not\equiv a_2 \pmod{p}$.*

The first direction is, as stated, proved by the same method as above, and the converse direction is quite straightforwardly proved by invoking Wolstenholme's Theorem and Theorem 1.

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