Let t be a given positive integer and let $\{r_i\}_{i\in\mathbb{N}}$ be a bounded sequence of non-zero integers and set $r = \max_i |r_i|$. Define L_n to be the least common multiple of all integers in the set $\{1, 2, .., n\}$, and set $X_n = L_n \sum_{i=1}^{n} \frac{r_i}{i}$. Let $2 = p_1 < p_2 < \ldots < p_z \leq \max(r, t)$ be the sequence of primes smaller than $m = \max(r, t)+1$, and let $p_1^{e_{1,n}} \cdots p_z^{e_{z,n}}$ be the prime decomposition of the largest divisor $d(n)$ of X_n, consisting only of primes smaller than m. Define $f(m)$ to be such that, for all $n \geq f(m)$ we have $|X_n| > n^z$ [1], and define m_0 to be equal to $\max(f(m), m^{2z+1})$. Lastly, define the half-open interval $I_0 = [m_0, 2m_0)$. We shall then prove the following Theorem:

Theorem 1. *There exists an integer $n \in I_0$ for which X_n is divisible by a prime larger than or equal to m.*

Proof. Note that proving Theorem 1 is equivalent to proving there exists an integer $n \in I_0$ for which $d_n < |X_n|$. To this end, we shall construct a sequence $m_0 = n_1 < n_2 < \ldots < n_{z+1} < 2m_0$, such that at least one of these n_j work. To start off, choose $n_1 = m_0$. Now, once we have defined n_j for some j with $1 \leq j \leq z$, if $d(n_j) < |X_{n_j}|$, we are done and we can stop. If, on the other hand, $d(n_j) = |X_{n_j}|$, then there exists a $\sigma(j) \in \{1, 2, \ldots, z\}$ such that for $p_{\sigma(j)}$ we have $p_{\sigma(j)}^{e_{\sigma(j),n_j}} > n_j$ [2], because $|X_{n_j}| > n_j^z$. Then, let $p_{\sigma(j)}^{k_j}$ be the largest power of $p_{\sigma(j)}$ smaller than or equal to $m^{2z+2-2j}$, and set n_{j+1} equal to the smallest integer larger than n_j such that $\frac{n_{j+1}}{\gcd(n_{j+1}, r_{j+1})}$ is divisible by $p_{\sigma(j)}^{k_j}$. Further, define the half-open interval $I_j = [n_{j+1}, n_{j+1} + p_{\sigma(j)}^{k_j})$. [3] We then claim (assuming $d(n_j) = |X_{n_j}|$ for all j with $1 \leq j \leq z$) the following things:

1. $I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_z$
2. $p_{\sigma(j)}^{e_{\sigma(j),n}} < n$ for all $n \in I_j$.
3. $\sigma(i) \neq \sigma(j)$ if $1 \leq i < j \leq z$.
4. $d(n_{z+1}) < |X_{n_{z+1}}|$.

Proof of 1. By definition we have the trivial inequality $n_j < n_{j+1}$. We also have $m^{2z+1-2j} < p_{\sigma(j)}^{k_j} \leq m^{2z+2-2j}$. And we know $n_{j+1} \leq n_j + (m-1)p_{\sigma(j)}^{k_j}$. From these inequalities we can prove $I_{j-1} \supset I_j$, for all $j \in \{2, \ldots, z\}$;

[1] Just assume for now that this exists.
[2] Of course, there can be more than one such prime. Just pick, say, the smallest.
[3] Note that the smallest member in I_j is n_{j+1}, not n_j.

$$
\begin{aligned}
n_j &< n_{j+1} \\
&< n_{j+1} + p_{\sigma(j)}^{k_j} \\
&\leq n_j + (m-1)p_{\sigma(j)}^{k_j} + p_{\sigma(j)}^{k_j} \\
&\leq n_j + m^{2z+3-2j} \qquad (1) \\
&= n_j + m^{2z+1-2(j-1)} \\
&< n_j + p_{\sigma(j-1)}^{k_{j-1}}
\end{aligned}
$$

To prove $I_0 \supset I_1$, use the above reasoning up to and including (1) for $j = 1$, and remember that $n_1 = m_0 \geq m^{2z+1}$.

Proof of 2. Let $n \in I_j$. Then we have:

$$
\begin{aligned}
X_n &= L_n \sum_{i=1}^{n} \frac{r_i}{i} \\
&= L_n \sum_{i=1}^{n_j} \frac{r_i}{i} + L_n \sum_{i=n_j+1}^{n_{j+1}-1} \frac{r_i}{i} + \frac{L_n r_{n_{j+1}}}{n_{j+1}} + L_n \sum_{i=n_{j+1}+1}^{n} \frac{r_i}{i} \\
&= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4
\end{aligned}
$$

By assumption, Σ_1 is divisible by a power of $p_{\sigma(j)}$ that exceeds $n_j > n/2$. By definition of n_{j+1}, we know that Σ_2 is divisible by the largest power of $p_{\sigma(j)}$ that divides $\frac{L_n}{p_{\sigma(j)}^{k_j-1}}$. Because $n < n_j + p_{\sigma(j)}^{k_j}$, every term in Σ_4 is also divisible by the largest power of $p_{\sigma(j)}$ that divides $\frac{L_n}{p_{\sigma(j)}^{k_j-1}}$. But Σ_3 is *not* divisible by the largest power of $p_{\sigma(j)}$ that divides $\frac{L_n}{p_{\sigma(j)}^{k_j-1}}$. So the whole sum (i.e. X_n) is *not* divisible by the largest power of $p_{\sigma(j)}$ that divides $\frac{L_n}{p_{\sigma(j)}^{k_j-1}}$. In particular, the power of $p_{\sigma(j)}$ that divides X_n is smaller than or equal to $n/2 < n$.

Proof of 3. Assume $1 \leq i < j \leq z$. By 2. we have $p_{\sigma(i)}^{e_{\sigma(i),n}} < n$ for all $n \in I_i$, where I_i is, by 1., a superset of I_{j-1}, which in turn contains, by definition, n_j. In particular, $p_{\sigma(i)}^{e_{\sigma(i),n_j}} < n_j$. While, on the other hand, $\sigma(j)$ was defined as a number for which we have that $p_{\sigma(j)}^{e_{\sigma(j),n_j}}$ is larger than n_j. And thus, $\sigma(i)$ and $\sigma(j)$ cannot be equal.

Proof of 4. By 3. we know that $\sigma = (\sigma(1), \ldots, \sigma(z))$ is a permutation of $(1, \ldots, z)$. By 1. we know that $n_{z+1} \in I_j$ for all $j \in \{1, \ldots, z\}$, which by 2. implies that $d(n_{z+1}) = p_1^{e_{1,n_{z+1}}} \cdots p_z^{e_{z,n_{z+1}}} < n_{z+1} \cdots n_{z+1} = {n_{z+1}}^z < |X_{n_{z+1}}|$.