ON THE NON-MONOTONICITY OF THE DENOMINATOR OF THE SUM OF CONSECUTIVE UNIT FRACTIONS

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Abstract

Let $u_{a,b}$ and $v_{a,b}$ be coprime positive integers such that $\dfrac{u_{a,b}}{v_{a,b}} = \displaystyle\sum_{i=a}^{b} \frac{1}{a}$. In [1, p. 34] the following questions are asked: Does there, for every fixed a, exist a b such that $v_{a,b} < v_{a,b-1}$? If so, what is the least such $b = b(a)$? In [2] the first of these questions is answered affirmatively and an upper bound of $b(a) \leq 6(a-1)$ is obtained for all $a > 1$. In the remarks of [2], the following conjecture is set forth: $\displaystyle\liminf_{a\to\infty} \frac{b(a)-a}{\log a} = c$ for some finite, positive constant c. This note is concerned with proving that conjecture, for a c with $\dfrac{1}{2} \leq c \leq 2$.

1. Main result and proof

We can immediately state our Main Theorem:

Theorem 1. $\dfrac{1}{2} \leq \displaystyle\liminf_{a\to\infty} \frac{b(a)-a}{\log a} \leq 2$

Proof. First we will prove the lower bound. Let ϵ be a given small positive constant, assume that a is large enough in terms of ϵ, and let b be any integer such that $a < b < a + (\frac{1}{2} - \epsilon)\log a$. Then we shall see that $v_{a,b} > v_{a,b-1}$. To achieve this, define $L_{a,b}$ to be the least common multiple of $\{a, a+1, .., b\}$ and set $X_{a,b} = L_{a,b} \displaystyle\sum_{i=a}^{b} \frac{1}{a}$. Then: $v_{a,b} = \dfrac{L_{a,b}}{\gcd(X_{a,b}, L_{a,b})}$. So $v_{a,b} > v_{a,b-1}$ precisely when $\dfrac{L_{a,b}}{L_{a,b-1}} > \dfrac{\gcd(X_{a,b}, L_{a,b})}{\gcd(X_{a,b-1}, L_{a,b-1})}$. We will show that this is indeed the case, by proving two inequalities: $\dfrac{L_{a,b}}{L_{a,b-1}} > \sqrt{b}$ and $\dfrac{\gcd(X_{a,b}, L_{a,b})}{\gcd(X_{a,b-1}, L_{a,b-1})} < \sqrt{b}$. This first inequality is established as follows:

$$
\begin{aligned}
\frac{L_{a,b-1}}{L_{a,b}} &= \frac{L_{a,b-1}}{\operatorname{lcm}(L_{a,b-1}, b)} \\
&= \frac{L_{a,b-1}\gcd(L_{a,b-1}, b)}{bL_{a,b-1}} \\
&= b^{-1}\gcd(L_{a,b-1}, b) \\
&= b^{-1}\prod_{p^k \le b-a < p^{k+1}} \gcd(p^k, b) \\
&\le b^{-1}\prod_{p^k \le (\frac{1}{2}-\epsilon)\log a < p^{k+1}} p^k \\
&= b^{-1}e^{(\frac{1}{2}-\epsilon+o(1))\log a} \qquad (1)\\
&< b^{-1}\sqrt{a} \\
&< b^{-1}\sqrt{b} \\
&= \frac{1}{\sqrt{b}}
\end{aligned}
$$

Where 1 is obtained as a consequence of the prime number theorem. For the second inequality, let p be a prime and let k be such that p^k exactly divides $L_{a,b-1}$. Then p can belong to three different sets:

1. The set of all p, such that p doesn't divide b.
2. The set of all p, such that p^{k+1} divides b.
3. The set of all p, such that $p^{k'}$ exactly divides b for some k' with $1 \le k' \le k$.

To deal with primes belonging to the first set, note that $\frac{L_{a,b}}{L_{a,b-1}} \not\equiv 0 \pmod{p}$, while $\frac{L_{a,b}}{b} \equiv 0 \pmod{p^k}$. We thus have:

$$
\begin{aligned}
X_{a,b} &= \frac{L_{a,b}X_{a,b-1}}{L_{a,b-1}} + \frac{L_{a,b}}{b} \\
&\equiv \frac{L_{a,b}X_{a,b-1}}{L_{a,b-1}} \pmod{p^k}
\end{aligned}
$$

And, since $\frac{L_{a,b}}{L_{a,b-1}} \not\equiv 0 \pmod{p}$, the largest power of p that divides $X_{a,b-1}$ is the same as the largest power of p that divides $X_{a,b}$. And this in turn implies that the largest power of p that divides $\gcd(X_{a,b}, L_{a,b})$ is the same as the largest power of p that divides $\gcd(X_{a,b-1}, L_{a,b-1})$. So these primes can be ignored in determining the size of $\dfrac{\gcd(X_{a,b}, L_{a,b})}{\gcd(X_{a,b-1}, L_{a,b-1})}$.

If p is a member of the second set we have: $\dfrac{L_{a,b}}{L_{a,b-1}} \equiv 0 \neq \dfrac{L_{a,b}}{b} \pmod{p}$, and so:

$$\begin{aligned} X_{a,b} &= \frac{L_{a,b} X_{a,b-1}}{L_{a,b-1}} + \frac{L_{a,b}}{b} \\ &\neq 0 \qquad\qquad\qquad\qquad\qquad\qquad \pmod{p} \end{aligned}$$

Thus, trivially, p doesn't divide $\gcd(X_{a,b}, L_{a,b})$ either. This, also trivially, implies that the power of p that divides $\gcd(X_{a,b-1}, L_{a,b-1})$ must be at least as large as the power of p that divides $\gcd(X_{a,b}, L_{a,b})$. So these primes can't be responsible either for making $\gcd(X_{a,b}, L_{a,b})$ larger than $\gcd(X_{a,b-1}, L_{a,b-1})$.

In conclusion, the only primes that can make $\gcd(X_{a,b}, L_{a,b})$ larger than $\gcd(X_{a,b-1}, L_{a,b-1})$ are the primes from the third set. So $\gcd(X_{a,b}, L_{a,b})$ can never be larger than $d \gcd(X_{a,b-1}, L_{a,b-1})$, where d is the largest divisor of b, composed solely of primes of the third set. Note that a prime p belongs to the third set, precisely when $p^{k'}$ exactly divides b and such that $b - p^{k'} \geq a$. In other words, d must be smaller than $\displaystyle\prod_{p^k \leq b-a < p^{k+1}} p^k$, which, as noted before, is smaller than $\sqrt{b}$, proving that we indeed have $\sqrt{b} > \dfrac{\gcd(X_{a,b}, L_{a,b})}{\gcd(X_{a,b-1}, L_{a,b-1})}$, which finishes up our proof of the lower bound of Theorem 1.

For the upper bound, let q' be the product of all primes between $\dfrac{3^k}{2}$ en 3^k and define $q = q'$ if $q' \equiv 2 \pmod 3$ and $q = 2q'$ if $q' \equiv 1 \pmod 3$. Note that, by the prime number theorem, $q = e^{(1/2+o(1))3^k}$. Now, choose $a = (q-1)3^k$ and set $b = a + 3^k$. We claim that $v_{a,b} < v_{a,b-1}$. Which implies that we have: $b(a) \leq b = a + 3^k = a + (2 + o(1)) \log a$. To prove that $v_{a,b} < v_{a,b-1}$, first observe that every prime power divisor of b is smaller than or equal to 3^k. And since $b - a = 3^k$, every prime power divisor of b is also a prime power divisor of some number between a and $b-1$ (inclusive). This implies that $L_{a,b} = L_{a,b-1}$. So to prove $v_{a,b} < v_{a,b-1}$, it suffices to show that $\gcd(X_{a,b}, L_{a,b}) > \gcd(X_{a,b-1}, L_{a,b-1})$. We now have to consider four distinct sets of primes:

1. The set of all p, such that p doesn't divide b and $p > 3$.
2. The set of all p, such that p divides b and $p > 3$.
3. $\{2\}$
4. $\{3\}$

Just like in our proof of the lower bound, no p from the first set has any influence on the relative sizes of $\gcd(X_{a,b}, L_{a,b})$ and $\gcd(X_{a,b-1}, L_{a,b-1})$.

For primes p from the second set, we have that $\frac{L_{a,b-1}}{i}$ vanishes modulo p, unless p divides i. But if $a \le i \le b-1$, then p divides i if, and only if, $i = b - p$, because $b - 2p < b - 3^k = a$. So we get:

$$\begin{aligned} X_{a,b-1} &= L_{a,b-1} \sum_{i=a}^{b} \frac{1}{i} \\ &\equiv \frac{L_{a,b-1}}{b-p} && \pmod{p} \\ &\neq 0 && \pmod{p} \end{aligned}$$

And we see that these primes also can't be responsible for making $\gcd(X_{a,b}, L_{a,b})$ smaller than $\gcd(X_{a,b-1}, L_{a,b-1})$.

For the prime 2, we use the well-known fact that among consecutive integers, there is exactly one that is divisible by a power of 2, such that no other number from that interval is also divisible by that power. In other words, $\frac{L_{a,b-1}}{i}$ vanishes modulo p, for all but one value of i. This implies that $X_{a,n}$ is always odd, in particular when $n = b - 1$.

All there is left is the prime 3. By the same method we used before, we see that 3 doesn't divide $X_{a,b-1}$;

$$\begin{aligned} X_{a,b-1} &= L_{a,b-1} \sum_{i=a}^{b} \frac{1}{i} \\ &\equiv \frac{L_{a,b-1}}{a} && \pmod{3} \\ &\neq 0 && \pmod{3} \end{aligned}$$

But 3 does divide $X_{a,b}$, thereby concluding our proof of the upper bound, and concluding this note:

$$
\begin{aligned}
X_{a,b} &= L_{a,b} \sum_{i=a}^{b} \frac{1}{i} \\
&\equiv \frac{L_{a,b}}{a} + \frac{L_{a,b}}{b} && \pmod 3 \\
&\equiv (2q-1)\frac{L_{a,b}}{q(q-1)3^k} && \pmod 3 \\
&\equiv 0 && \pmod 3
\end{aligned}
$$

□

References

[1] P. Erdös, R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*. Enseign. Math. (2), vol. 28, Enseignement Math., Geneva, 1980. Also available here.

[2] W. van Doorn, *Sums of consecutive (generalized) unit fractions*. Available here.