SUMS OF CONSECUTIVE (GENERALIZED) UNIT FRACTIONS

Wouter van Doorn
Undergraduate at Utrecht University, Utrecht, The Netherlands
wonterman1@hotmail.com

Received: , Revised: , Accepted: , Published:

Abstract

Let L_n be the least common multiple of $\{1, 2, .., n\}$ and X_n such that $\frac{X_n}{L_n} = \sum_{i=1}^{n} \frac{1}{i}$. Let $\sum_{i=a+1}^{b} \frac{1}{i} = \frac{u_{a,b}}{v_{a,b}}$ with $u_{a,b}$ and $v_{a,b}$ coprime. In their influential monograph [1, p. 34], Erdös and Graham ask, among many others, the following questions: Does $\gcd(X_n, L_n) > 1$ happen infinitely often? Does there, for every fixed a, exist a b such that $v_{a,b} < v_{a,b-1}$? If so, what is the least such $b = b(a)$? In this note we will investigate these issues in a more general setting, answer the first two questions in the affirmative and obtain an upper bound of $b(a) \leq 6a$, for all $a \geq 1$.

1. Notation and Definitions

In this note, L_n will denote the least common multiple of $\{1|s_1|, 2|s_2|, .., n|s_n|\}$ and X_n will be defined as $L_n \sum_{i=1}^{n} \frac{r_i}{is_i}$, where $\{r_i\}_{i\in\mathbb{N}}$ and $\{s_i\}_{i\in\mathbb{N}}$ are given periodic sequences of integers. That is, for all i we have $r_{i+t} = r_i$ for some $t \in \mathbb{N}$ and $s_{i+t'} = s_i$ for some $t' \in \mathbb{N}$. We will, without loss of generality, assume $t = t'$ throughout and set $s^* = \max_i |s_i|$ and $r^* = \max_i |r_i|$, where it is assumed that $r^* \geq 1$, to avoid the trivial case. We will also define $X_{0,b} = X_b$ and, for $a \geq 1$, define $X_{a,b} = X_b - \frac{L_b X_a}{L_a}$ and set $g_{a,b} = \gcd(L_b, X_{a,b})$. $u_{a,b}$ and $v_{a,b}$ are coprime integers such that $v_{a,b}$ is positive and $\frac{u_{a,b}}{v_{a,b}} = \sum_{i=a+1}^{b} \frac{r_i}{is_i}$. Furthermore, $\lambda = \lambda(t)$ is defined as the smallest positive integer such that, for all p coprime to t, we have $p^\lambda \equiv 1 \pmod{t}$ while, of course, Euler's Totient Theorem implies that λ exists. Last, but not least, $b(a)$ is defined as the smallest integer larger than a such that $v_{a,b} < v_{a,b-1}$, if this exists, and $b(a) = \infty$ otherwise. The main goal of this paper is proving that, in certain cases, $b(a)$ is finite. The letter p is reserved for prime numbers and every other letter used will, unless stated otherwise, always denote an integer (usually non-negative). Furthermore, $x|y$ reads 'x divides y', $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N}$ for the set of positive integers.

2. Main Results and Proofs

Theorem 1. *If a positive integer l and a prime $p > \max(l|s_l|, r^*, s^*, t)$ exist, such that $r_l \neq 0$ and p divides X_l, then infinitely many b exist with: $v_{a,b} < v_{a,b-1}$. Furthermore, for $b(a)$ we then have the following linear upper bound: $b(a) \leq \max(c, ca)$, where $c = lp^\lambda$.*

Proof. Note that, for $r_1 = s_1 = t = 1$ we may use $p = 3, l = 2$, as $3|X_2$ in that case. So Theorem 1 implies the claimed upper bound $b(a) \leq \max(6, 6a)$ in the special case $r_1 = s_1 = t = 1$. In general we have:

$$\begin{aligned} \frac{u_{a,b}}{v_{a,b}} &= \sum_{i=a+1}^{b} \frac{r_i}{is_i} \\ &= \sum_{i=1}^{b} \frac{r_i}{is_i} - \sum_{i=1}^{a} \frac{r_i}{is_i} \\ &= \frac{X_b}{L_b} - \frac{X_a}{L_a} \\ &= \frac{X_b - X_a * \frac{L_b}{L_a}}{L_b} \\ &= \frac{X_{a,b}}{L_b} \end{aligned}$$

where $v_{a,b} = \dfrac{L_b}{g_{a,b}}$. So if $L_b = L_{b-1}$, then $v_{a,b} < v_{a,b-1}$ holds true, precisely when $g_{a,b} > g_{a,b-1}$. Now we will introduce the most important Lemma of this paper, which will be used again and again. To be able to state it, let c be such that, for all i, p^{c+1} doesn't divide s_i and also such that p^{c+1} doesn't divide t. We then have:

Lemma 1. *Let k and m be any non-negative integers. If p^{2c+m} divides n and n' is any integer with $np^{\lambda k} \leq n' < (n+1)p^{\lambda k}$, then: $p^{m+1}|X_n$ if, and only if, $p^{m+1}|X_{n'}$.* [1]

[1] Apart from the proof of Lemma 5, we only need the case $c = m = 0$.

Proof. First we will prove that $p^{m+1}|X_n$ if, and only if, $p^{m+1}|X_{np^{\lambda k}}$. Then we will show that if $np^{\lambda k} < n' < (n+1)p^{\lambda k}$, then $p^{m+1}|X_{n'}$ and $p^{m+1}|X_{n'-1}$ are equivalent. The proof is mainly based on the observation that if $\frac{L_n r_i}{is_i}$ doesn't vanish modulo p^{m+1}, then, since p^{c+1} doesn't divide s_i, i must divisible by the power of p that divides $L_n p^{-(c+m)}$. In particular, i must then be divisible by the power of p that divides $np^{-(c+m)}$, which is at least p^c. This implies:

$$\begin{aligned} X_n &= L_n \sum_{i=1}^{n} \frac{r_i}{is_i} \\ &\equiv L_n \sum_{i=1}^{np^{-c}} \frac{r_{ip^c}}{ip^c s_{ip^c}} && \pmod{p^{m+1}} \\ &\equiv \frac{L_n}{p^c} \sum_{i=1}^{np^{-c}} \frac{r_{ip^c}}{is_{ip^c}} && \pmod{p^{m+1}} \end{aligned}$$

And similarly:

$$\begin{aligned} X_{np^{\lambda k}} &= L_{np^{\lambda k}} \sum_{i=1}^{np^{\lambda k}} \frac{r_i}{is_i} \\ &\equiv L_{np^{\lambda k}} \sum_{i=1}^{np^{-c}} \frac{r_{ip^{\lambda k+c}}}{ip^{\lambda k+c} s_{ip^{\lambda k+c}}} && \pmod{p^{m+1}} \\ &\equiv \frac{L_{np^{\lambda k}}}{p^{\lambda k+c}} \sum_{i=1}^{np^{-c}} \frac{r_{ip^c}}{is_{ip^c}} && \pmod{p^{m+1}} \end{aligned}$$

So to prove that $p^{m+1}|X_n$ if, and only if, $p^{m+1}|X_{np^{\lambda k}}$, it suffices to show that the largest power of p that divides $\frac{L_n}{p^c}$ is equal to the largest power of p that divides $\frac{L_{np^{\lambda k}}}{p^{\lambda k+c}}$. Or, equivalently, it suffices to show that the largest power of p that divides L_n is equal to the largest power of p that divides $\frac{L_{np^{\lambda k}}}{p^{\lambda k}}$. To see that it can't be larger, let $j \le n$ be such that the largest power of p that divides js_j equals the largest power of p that divides L_n. Note that, by the fact that $p^{2c}|n$, j must then be divisible by p^c, which implies that $s_{jp^{\lambda k}} = s_j$. And thus, since $jp^{\lambda k} \le np^{\lambda k}$, $L_{np^{\lambda k}}$ must be divisible by $jp^{\lambda k} s_{jp^{\lambda k}} = jp^{\lambda k} s_j$. To see that it can't be smaller, we use the same reasoning in reverse; let $j \le np^{\lambda k}$ be such that the largest power of p that divides js_j equals the largest power of p that divides $L_{np^{\lambda k}}$. Note that j must now be divisible by $p^{\lambda k+c}$, which implies that $s_{jp^{-\lambda k}}$ exists and must equal s_j. And thus, L_n must be divisible by $jp^{-\lambda k}s_{jp^{-\lambda k}} = jp^{-\lambda k}s_j$. And this concludes our proof of the fact that $p^{m+1}|X_n$ and $p^{m+1}|X_{np^{\lambda k}}$ are equivalent.

Now, let n' be such that $np^{\lambda k} < n' < (n+1)p^{\lambda k} = np^{\lambda k} + p^{\lambda k}$. We will now prove that $p^{m+1}|X_{n'}$ and $p^{m+1}|X_{n'-1}$ are equivalent, finishing our proof of Lemma 1. To do this, note that, in general, $X_{n'} = \frac{X_{n'-1}L_{n'}}{L_{n'-1}} + \frac{L_{n'}r_{n'}}{n's'_n}$. First we will show that $\frac{L_{n'}r_{n'}}{n's'_n}$ vanishes modulo p^{m+1}, implying that we have: $X_{n'} \equiv \frac{X_{n'-1}L_{n'}}{L_{n'-1}}$ (mod p^{m+1}). Of course, $L_{n'}$ is divisible by $np^{\lambda k}$, and thus in particular by $p^{2c+m+\lambda k}$. While on the other hand, n' is not divisible by $p^{\lambda k}$, which shows us that $n's_{n'}$ is not divisible by $p^{c+\lambda k}$, so $\frac{L_{n'}r_{n'}}{n's'_n}$ must be divisible by p^{c+m+1} and, in particular, vanishes modulo p^{m+1}. So, indeed, $X_{n'} \equiv \frac{X_{n'-1}L_{n'}}{L_{n'-1}} \pmod{p^{m+1}}$. Furthermore, this suffices if p doesn't divide $\frac{L_{n'}}{L_{n'-1}}$, because then $X_{n'} \equiv 0 \pmod{p^{m+1}}$ if, and only if, $X_{n'-1} \equiv 0 \pmod{p^{m+1}}$. And indeed we will see that p doesn't divide $\frac{L_{n'}}{L_{n'-1}}$, by using basically the same reasoning we used above. Because if $np^{\lambda k} < n' < np^{\lambda k} + p^{\lambda k}$, then, again, n' is not divisible by $p^{\lambda k}$, which implies that $n's_{n'}$ is, again, not divisible by $p^{c+\lambda k}$. While on the other hand, $L_{n'-1}$ is still divisible by $np^{\lambda k}$, and in particular by $p^{2c+m+\lambda k}$, which is thus larger than the largest power of p that divides $n's_{n'}$. So the largest power of p that divides $L_{n'} = \text{lcm}(L_{n'-1}, n's_{n'})$ is the same as the largest power of p that divides $L_{n'-1}$. □

With the help of Lemma 1 we can begin our construction of infinitely many b for which $v_{a,b} < v_{a,b-1}$. To that end, let p be a prime larger than $\max(l|s_l|, r^*, s^*, t)$, that divides X_l and note that $l > 1$, because $p > r^* \geq |X_1|$. Set $b = lp^{\lambda k}$, such that $p^{\lambda k} > a'$, where $a' = \max(a, 1)$. Now, by Lemma 1, $p|X_b$. We also have $L_b = L_{b-1}$, because $L_b = \text{lcm}(bs_b, L_{b-1}) = \text{lcm}(lp^{\lambda k}s_b, L_{b-1}) = L_{b-1}$, since $\gcd(p^{\lambda k}, ls_b) = 1$, while $p^{\lambda k} \leq b-1$ and also $l|s_b| = l|s_l| < p \leq b-1$. So to prove the first part of Theorem 1, it suffices to show $g_{a,b} > g_{a,b-1}$. For the second part, observe that for the smallest possible k, we have $p^{\lambda k - \lambda} \leq a'$, and thus $b = lp^{\lambda k} \leq a'lp^{\lambda}$.

Now, since $b = lp^{\lambda k} > p^{\lambda k} > a$, there exists a power of p between a and b, implying that p divides $\frac{L_bX_a}{L_a}$. Since we also have that p divedes X_b, we know that $p|X_{a,b}$, too. While on the other hand, since $\frac{L_br_b}{bs_b}$ doesn't vanish modulo p, p doesn't divide X_{b-1} and, by implication, doesn't divide $X_{a,b-1}$ either. To be able to finish our proof of Theorem 1, we shall need the following Lemma:

Lemma 2. *Let a, b, c, d be integers such that c divides ab. Then:* $\gcd(a, d)$ *divides* $\gcd(c, d)\gcd(a, \frac{ab}{c} + d)$*, which in turn divides* $c\gcd(a, \frac{ab}{c} + d)$

Proof. The second part is trivial. To prove the first part, let p be any prime dividing a and let $p^\alpha, p^\beta, p^\gamma, p^\delta$ be the largest powers of p dividing a, b, c, d respectively (note: $\alpha+\beta \geq \gamma$). If $\min(\gamma, \delta) = \gamma$, the largest power of p that divides $\gcd(c, d)\gcd(a, \frac{ab}{c} + d)$ equals:

$$\begin{aligned}\gcd(p^{\alpha+\min(\gamma,\delta)}, p^{\alpha+\beta+\min(\gamma,\delta)-\gamma} + p^{\delta+\min(\gamma,\delta)}) &\geq \gcd(p^\alpha, p^{\alpha+\beta} + p^{\delta+\gamma}) \\ &= \gcd(p^\alpha, p^{\delta+\gamma}) \\ &\geq \gcd(p^\alpha, p^\delta)\end{aligned}$$

which equals the largest power of p that divides $\gcd(a, d)$, which we wanted to prove. And if $\min(\gamma, \delta) = \delta$, the largest power of p that divides $\gcd(c, d)\gcd(a, \frac{ab}{c} + d)$ equals:

$$\begin{aligned}\gcd(p^{\alpha+\min(\gamma,\delta)}, p^{\alpha+\beta+\min(\gamma,\delta)-\gamma} + p^{\delta+\min(\gamma,\delta)}) &\geq \gcd(p^\alpha, p^{\alpha+\beta+\delta-\gamma} + p^{2\delta}) \\ &\geq \gcd(p^\alpha, p^\delta + p^{2\delta}) \\ &\geq \gcd(p^\alpha, p^\delta)\end{aligned}$$

which, again, equals the largest power of p that divides $\gcd(a, d)$, which proves Lemma 2. □

Combining our knowledge of Lemma 2 and the fact that p divides $X_{a,b}$, but doesn't divide $X_{a,b-1}$, finishes up our proof for Theorem 1:

$$
\begin{aligned}
g_{a,b} &= \gcd\left(L_b, X_{a,b}\right) \\
&= \gcd\left(p^{\lambda k}, X_{a,b}\right) * \gcd\left(\frac{L_b}{p^{\lambda k}}, X_{a,b}\right) \\
&\geq p \gcd\left(\frac{L_b}{p^{\lambda k}}, X_{a,b}\right) \\
&= p \gcd\left(\frac{L_b}{p^{\lambda k}}, \frac{L_b r_b}{b s_b} + X_{a,b-1}\right) \\
&= p \gcd\left(\frac{L_b}{p^{\lambda k}}, \frac{L_b}{p^{\lambda k}} \frac{r_b}{l s_b} + X_{a,b-1}\right) \\
&\geq \frac{p}{l|s_b|} \gcd\left(\frac{L_b}{p^{\lambda k}}, X_{a,b-1}\right) \\
&= \frac{p}{l|s_l|} \gcd\left(\frac{L_b}{p^{\lambda k}}, X_{a,b-1}\right) \\
&> \gcd\left(\frac{L_b}{p^{\lambda k}}, X_{a,b-1}\right) \\
&= \gcd\left(L_b, X_{a,b-1}\right) \\
&= g_{a,b-1}
\end{aligned}
$$

□

Note that, by the same reasoning, we can actually show that $ls_b g_{a,b}$ is a multiple of $pg_{a,b-1}$. We can now go on to a different case, where there doesn't necessarily exist l and $p > \max(l|s_l|, r^*, s^*, t)$, such that $p|X_l$. But not before we conjecture the following, which, if true, would imply that Theorem 1 is completely general:

Conjecture 1. *For every $c \in \mathbb{R}$ there exist l and p such that $p > cl$ and $p|X_l$.*

But, alas, Conjecture 1 may be out of reach for the moment. So we proceed along a different path, assuming (the need for this will be clear later) the following for the rest of this paper:

Conjecture 2. *(Schanuel's Conjecture) If $z_1, .., z_m$ are rationally independent reals, then at least m of the following are algebraically independent: $z_1, .., z_m, e^{z_1}, .., e^{z_m}$.*

Now, by the above argument, we still do know that if a large p divides X_b and doesn't divide X_{b-1} then, assuming a power of p lies between a and b, we have that $pg_{a,b-1}$ divides $ls_b g_{a,b}$. But this doesn't imply $g_{a,b} > g_{a,b-1}$, if we are not allowed to assume $p > l|s_b|$. So our plan is as follows: we want to show that $\gcd(ls_b, X_{a,b-1})$ divides some small enough constant R, say, implying with the use of Lemma 2 that we actually have that $pg_{a,b-1}$ divides $Rg_{a,b}$. And, of course, if p is larger than R, this suffices. To carry out this plan, we thus need two things: we need a large prime p dividing X_b and we need $\gcd(ls_b, X_{a,b-1})$ to be small. Let's start with finding such a large prime p.

Theorem 2. *If Conjecture 2 holds and $r_t \neq 0$, then there exist arbitrarily large primes p, such that $p|X_n$, for some $n = lp^k$, where $1 < l < p$.*

Proof. To prove this, we first have to show that X_n has prime divisors to begin with;

Lemma 3. *Let R be any positive integer. Then for all $n > \max(R, s^*, t, 10)$, at least one of $X_n, X_{n+1}, .., X_{n+t}$ is larger than R.*

Proof. Let n be larger than $\max(R, s^*, t, 10)$ and $j \in \{1, 2, .., t\}$ such that $r_j \neq 0$. We then have for two consecutive partial sums:

$$\begin{aligned}
|X_{n+j-1}| + |X_{n+j}| &\geq L_{n+j-1}\left|\sum_{i=1}^{n+j-1}\frac{r_i}{is_i}\right| + L_{n+j}\left|\sum_{i=1}^{n+j}\frac{r_i}{is_i}\right| \\
&\geq L_n\left(\left|\sum_{i=1}^{n+j-1}\frac{r_i}{is_i}\right| + \left|\sum_{i=1}^{n+j}\frac{r_i}{is_i}\right|\right) \\
&\geq L_n\left|\sum_{i=1}^{n+j}\frac{r_i}{is_i} - \sum_{i=1}^{n+j-1}\frac{r_i}{is_i}\right| \\
&= L_n\left|\frac{r_{n+j}}{(n+j)s_{n+j}}\right| \\
&> \frac{L_n}{2n^2} \\
&> 2R
\end{aligned}$$

So at least one of them is larger than R. □

So for infinitely many n, we have that X_n is large and, thus, must have prime divisors. Now we need to show that these prime divisors themselves must be large sometimes, in which the following Lemma and the Lemma thereafter will be instrumental.

Lemma 4. *For all $p > \max(r^*, s^*, t)$, there exists a positive integer c_p, such that for all $k \in \mathbb{N}$ and all n with: $c_p p^{\lambda k} \leq n < (c_p + 1)p^{\lambda k}$, we have that p doesn't divide X_n.*

Proof. Set c_p to be the smallest integer such that $X_{c_p} \neq 0$. Or, equivalently, such that $r_{c_p} \neq 0$, which must happen for some $c_p \leq t$. So we then have that $X_{c_p} =$

$\frac{L_{c_p} r_{c_p}}{c_p s_{c_p}}$, which has all its prime divisors smaller than or equal to $\max(r^*, s^*, t)$. So if p is larger than that, p doesn't divide X_{c_p}. By Lemma 1 we have as a consequence that p doesn't divide X_n for all k and all n with: $c_p p^{\lambda k} \leq n < (c_p + 1)p^{\lambda k}$, which proves Lemma 4. □

Now we will state and prove a Lemma similar to Lemma 4, that partly deals with all small p:

Lemma 5. *If $r_t \neq 0$, then there exists a positive integer R, such that for all p, there exists a positive integer c_p, such that for all $k \in \mathbb{N}$ and all n with: $c_p p^{\lambda k} \leq n < (c_p + 1)p^{\lambda k}$, we have that $p^m | X_n$ implies $p^m | R$.*[2]

Proof. The reason we need this refinement, is that for small p it is unfortunately not necessary that infinitely many values of n exist for which $\gcd(p, X_n) = 1$. [3] But, assuming $r_t \neq 0$, we are able to find large intervals, such that for all n inside these intervals, the largest power of p that divides X_n is bounded. [4] Set $n = (d + ps^*t)p^{13r^*s^*t}$, where d is the largest divisor of t, that is not divisible by p. Then $r_n = r_t \neq 0$. We also claim that $L_n = L_{n-1}$. To see this, note that $d + ps^*t$ is not divisible by p. So it suffices to show that both $(d + ps^*t)s_n$ and $p^{13r^*s^*t}s_n$ divide L_{n-1}. But both of these are smaller than n, so this is immediate. Also, $n = (d + ps^*t)p^{13r^*s^*t} < 2ps^*tp^{13r^*s^*t} < p^{3s^*t}p^{13r^*s^*t} \leq p^{16r^*s^*t}$, so, since for all i, p^{s^*} doesn't divide s_i, the largest power of p that divides L_n is smaller than $p^{16r^*s^*t}p^{s^*} \leq p^{17r^*s^*t}$, which in turn implies that the largest power of p that divides $L_n r_n$ is smaller than $p^{17r^*s^*t}p^{r^*} \leq p^{18r^*s^*t}$. And since n is divisible by $p^{13r^*s^*t}$, we have:

$$X_n = X_{n-1} + \frac{L_n r_n}{n s_n} \not\equiv X_{n-1} \pmod{p^{5r^*s^*t}}$$

So at least one of X_{n-1} and X_n is not divisible by $p^{5r^*s^*t}$. If X_n is not divisible by $p^{5r^*s^*t}$, we can invoke Lemma 1, and thereby proving Lemma 5 with $c_p = (d+ps^*t)p^{13r^*s^*t}$. If X_{n-1} is not divisible by $p^{5r^*s^*t}$, then $X_{n-p^{7r^*s^*t}}$ is not divisible by $p^{5r^*s^*t}$ either, since $\frac{L_{n'} r_{n'}}{n' s_{n'}}$ vanishes modulo $p^{5r^*s^*t}$ for n' with: $n - p^{7r^*s^*t} < n' < n$. And this implies that we can invoke Lemma 1 again, proving Lemma 5 with $c_p = (d + ps^*t)p^{13r^*s^*t} - p^{7r^*s^*t}$. □

With the help of Kronecker's Approximation Theorem we will soon see that we can extend Lemma 4 and Lemma 5 to arbitrary sets of primes, which, combined with

[2] Thus, R may depend on the r_i, s_i and t (which are assumed to be given), but is independent of everything else.

[3] Even assuming $\gcd(r_1, r_2, .., r_t) = 1$. For example, if $r_1 = s_1 = s_2 = 1$ and $r_2 = t = 2$, then 2 divides X_n for all $n \geq 2$.

[4] When $r_t = 0$, this may very well fail. Consider for example $r_1 = s_1 = s_2 = 1, r_2 = 0$ and $t = 2$. In this case, every interval of size 2^k contains an n, such that $2^{2k-2} | X_n$.

Lemma 3, implies that X_n has arbitrarily large prime divisors (if $r_t \neq 0$). So let $2 = p_1 < p_2 < .. < p_m$ be the sequence of primes up to some large arbitrary bound (for concreteness' sake, think of this bound as something larger than $\max(R, r^*, s^*, t)$), define $P = p_m!$ and let c_{p_j} be the constant from Lemma 4 or Lemma 5 corresponding to the prime p_j. Now, if for some k and n and all j with: $1 \leq j \leq m$, the following holds:

$$c_{p_j} p_j^{\lambda k} < n < (c_{p_j} + 1) p_j^{\lambda k} - (t+1) \tag{1}$$

then $\gcd(X_{n+i}, P) \leq R$, for all i with $-1 \leq i \leq t$ [5], while, by Lemma 3 at least one of $X_n, X_{n+1}, .., X_{n+t}$ is larger than R and must therefore be divisible by some prime larger than our arbitrary bound. And indeed, we will see that there exist such k and n. First we need to define the following constants:

$$\alpha_j = \frac{1}{\lambda \log p_j}$$

$$\beta_j = \frac{\log c_{p_j} + \log\left(c_{p_j} + 1 - \frac{t+1}{p_j^{t+1}}\right)}{-2\lambda \log p_j}$$

$$\epsilon_j = \frac{\log\left(c_{p_j} + 1 - \frac{t+1}{p_j^{t+1}}\right) - \log c_{p_j}}{2\lambda \log p_j}$$

These constants are just defined so that we will end up with manageable expressions. The only important thing to note is that $\epsilon_j > 0$, for all j. Now, if there exists some n and a $k \geq t+1$ such that the following equations are satisfied, then (1) is also satisfied.

[5]the reason we also want to include X_{n-1}, is that, eventually, we want to prove something like $g_{n-1} < g_n$, and thus g_{n-1} must be as small as possible, which in this case means: have a small gcd with P.

$$
\begin{aligned}
c_{p_j} p_j^{\lambda k} &< n < (c_{p_j}+1)p_j^{\lambda k} - (t+1) \\
c_{p_j} p_j^{\lambda k} &< n < p_j^{\lambda k}\left(c_{p_j}+1-\frac{t+1}{p_j^{\lambda k}}\right) \\
c_{p_j} p_j^{\lambda k} &< n < p_j^{\lambda k}\left(c_{p_j}+1-\frac{t+1}{p_j^{t+1}}\right) \\
\lambda k \log p_j + \log c_{p_j} &< \log n < \lambda k \log p_j + \log\left(c_{p_j}+1-\frac{t+1}{p_j^{t+1}}\right) \\
k &< \frac{\log n}{\lambda \log p_j} + \frac{\log c_{p_j}}{-\lambda \log p_j} < k + \frac{\log\left(c_{p_j}+1-\frac{t+1}{p_j^{t+1}}\right) - \log c_{p_j}}{\lambda \log p_j} \\
k &< \alpha_j \log n + \beta_j + \epsilon_j < k + 2\epsilon_j \\
||\alpha_j \log n + \beta_j|| &< \epsilon_j \qquad (2)
\end{aligned}
$$

where $||x||$ denotes the distance from x to the nearest integer. We claim that the m equations defined by (2) can oftentimes be simultaneously solved. [6] In fact, we can even get infinitely many n of a specific form that solve all m equations defined by (2). More precisely, we will prove that for all primes $p > p_m$ and all integers l, there exist infinitely many n of the form $lp^{\lambda k'}$ such that (2) holds for all $1 \le j \le m$ (note that k' is now variable). To prove this somewhat stronger statement, we'll use the following definitions:

$$
\begin{aligned}
\alpha_j' &= \alpha_j \lambda \log p = \frac{\log p}{\log p_j} \\
\beta_j' &= \alpha_j \log l + \beta_j = \frac{2\log l - \log c_{p_j} - \log\left(c_{p_j}+1-\frac{t+1}{p_j^{t+1}}\right)}{2\lambda \log p_j} \\
\epsilon &= \min_{1\le j\le m} \epsilon_j = \min_{1\le j\le m}\left(\frac{\log\left(c_{p_j}+1-\frac{t+1}{p_j^{t+1}}\right) - \log c_{p_j}}{2\lambda \log p_j}\right)
\end{aligned}
$$

These are again only defined to make sure we end up with manageable expressions. Again, $\epsilon > 0$, since we noted that $\epsilon_j > 0$ for all j. is So if we can satisfy the following

[6]Keep in mind that being able to solve these equations, implies having a positive integer n such that, for all i with $-1 \le i \le t$, $\gcd(X_{n+i}, P) \le R$

equations infinitely often, we found infinitely many (and thus large enough) n of the form $lp^{\lambda k'}$ such that $\gcd(X_{n+i}, P) \leq R$ for all i with: $-1 \leq i \leq t$:

$$\begin{aligned} ||\alpha_j \log n + \beta_j|| &< \epsilon_j \\ ||\alpha_j \log(lp^{\lambda k'}) + \beta_j|| &< \epsilon_j \\ ||\alpha_j \lambda k' \log p + \alpha_j \log l + \beta_j|| &< \epsilon_j \\ ||\alpha'_j k' + \beta'_j|| &< \epsilon \end{aligned} \tag{3}$$

By Kroneckers Approximation Theorem, there exist infinitely many k' such that the m equations defined by (3) are simultaneously satisfied, if the α'_j together with 1 are rationally independent. And this is where Conjecture 2 comes in:

Lemma 6. *Assume Conjecture 2 holds. Then the m terms $\dfrac{\log p}{\log p_j}$ are, together with* 1*, rationally independent.*

Proof. [7] Since by unique factorization we have that all the terms $\log p_i$, are rationally independent, Conjecture 2 shows that they must in fact be algebraically independent, which immediately gives the result. □

So we found more than t consecutive X_i for which $\gcd(X_i, P) \leq R$, while at least one of these X_i is larger than R and so we see that that X_i must have a prime divisor, say p, that doesn't divide P. This implies, of course, that p must be larger than our arbitrarily large bound, which, again, should be thought of as larger than $\max(R, r^*, s^*, t)$.

Now that we have shown that X_n has arbitrarily large prime divisors, we can finish up our proof of Theorem 2, by showing that if $n = lp^k$ is the smallest positive integer for which p divides X_n, and such that $\gcd(l, p) = 1$, then $1 < l < p$. First we will handle the first inequality, and note that this implies that p doesn't divide $\dfrac{L_n}{L_{n-1}}$, as we assume p is large, i.e. $p > s^*$. But if l is equal to 1, we immediately obtain a contradiction, since $r_n \not\equiv 0 \pmod p$ and, as we have noticed before, p divides $\dfrac{L_n r_i}{i s_i}$, unless $p^k | i$;

[7] this proof was shown to the author on www.mathoverflow.net, by Qiaochu Yuan and Matt Papanikolas

$$\begin{aligned} X_n &= L_n \sum_{i=1}^{n} \frac{r_i}{is_i} \\ &\equiv \frac{L_n r_n}{ns_n} && \pmod p \\ &\not\equiv 0 && \pmod p \end{aligned}$$

So $l > 1$. Second of all, since n is the smallest number with $p|X_n$, we have:

$$\begin{aligned} 0 &\equiv X_n && \pmod p \\ &= \frac{L_n X_{n-1}}{L_{n-1}} + \frac{L_n r_n}{ns_n} && \pmod p \\ &\not\equiv \frac{L_n r_n}{ns_n} && \pmod p \end{aligned}$$

Where the last inequality holds, because if equality would have held, then p would divide $\frac{L_n X_{n-1}}{L_{n-1}}$, which is impossible, by our proof that $l > 1$. So $\frac{L_n r_n}{ns_n}$ may not vanish modulo p, implying that the largest power of p that divides L_n must also divide n, which shows us that $l < p$. □

Now we are ready to prove, although still assuming Schanuel's Conjecture and $r_t \neq 0$, and, because of the ineffectiveness of Kroneckers Approximation Theorem, without an explicit upper bound, the finiteness of $b(a)$ in more generality:

Theorem 3. *If Conjecture 2 holds and $r_t \neq 0$, then there exist infinitely many b such that $v_{a,b} < v_{a,b-1}$.*

Proof. By Theorem 2, we have at our disposal a large prime p [8] such that $p|X_n$, for some $n = lp^k$. We can now go to the same process of extending Lemma 4 and Lemma 5 again, this time with all the prime divisors of ls_n, and find infinitely many b (we will now use b again for a generic number) of the form $np^{\lambda k'} = lp^{\lambda k' + k}$ such that $\gcd(X_{b-1}, ls_n) \leq R$. In particular, we can find infinitely many such b such that lps_n divides $\frac{L_b X_a}{L_a}$ (clearly all $b \geq L_a lps_n$ have this property), which implies $\gcd(X_{a,b-1}, ls_n) \leq R$, as well. We shall work with one such b (thereby fixing b) and show that we then have $v_{a,b} < v_{a,b-1}$, concluding our proof of Theorem 3. First, p divides X_b, by virtue of Lemma 1. Secondly, since p divides $\frac{L_b X_a}{L_a}$, p must also divide $X_{a,b}$. While on the other hand $\frac{L_b r_b}{bs_b} \not\equiv 0 \pmod p$, implying that

[8] We shall assume it to be larger than $\max(R, r^*, s^*, t)$.

p doesn't divide X_{b-1} and thus also doesn't divide $X_{a,b-1}$. Thirdly, we see that $L_b = L_{b-1}$, just like in our proof of Theorem 1 (which, again, implies that proving $g_{a,b-1} < g_{a,b}$ suffices), by noting that $\gcd(p^{\lambda k'+k}, ls_b) = 1$, while $p^{\lambda k'+k} \leq b-1$ and also $l|s_b| \leq ls^* < lp \leq b-1$. Putting this all together, and invoking Lemma 2 once more, finishes up our argument:

$$\begin{aligned}
g_{a,b} &= \gcd\left(L_b, X_{a,b}\right)\\
&= \gcd\left(p^{\lambda k'+k}, X_{a,b}\right) * \gcd\left(\frac{L_b}{p^{\lambda k'+k}}, X_{a,b}\right)\\
&\geq p\gcd\left(\frac{L_b}{p^{\lambda k'+k}}, X_{a,b}\right)\\
&= p\gcd\left(\frac{L_b}{p^{\lambda k'+k}}, \frac{L_b r_b}{b s_b} + X_{a,b-1}\right)\\
&= p\gcd\left(\frac{L_b}{p^{\lambda k'+k}}, \frac{L_b r_b}{l s_b p^{\lambda k'+k}} + X_{a,b-1}\right)\\
&\geq \frac{p}{\gcd(X_{a,b-1}, ls_b)}\gcd\left(\frac{L_b}{p^{\lambda k'+k}}, X_{a,b-1}\right)\\
&\geq \frac{p}{\gcd(X_{a,b-1}, ls_n)}\gcd\left(\frac{L_b}{p^{\lambda k'+k}}, X_{a,b-1}\right)\\
&\geq \frac{p}{R}\gcd\left(\frac{L_b}{p^{\lambda k'+k}}, X_{a,b-1}\right)\\
&> \gcd\left(\frac{L_b}{p^{\lambda k'+k}}, X_{a,b-1}\right)\\
&= \gcd\left(L_b, X_{a,b-1}\right)\\
&= g_{a,b-1}
\end{aligned}$$

□

3. Remarks

Clearly, it would be nice to prove Conjecture 1 (not to mention Conjecture 2!) so that, in general, we get a linear upper bound $b(a) \leq ca$ for some constant c. On the other hand, as far as the author knows, other bounds on $b(a)$ are not known either; Can we find a non-trivial function f, such that $\limsup_{a\to\infty} \frac{b(a)-a}{f(a)} > 0$? Also, it is an amusing observation that, for example, $\liminf_{a\to\infty} \frac{b(a)-a}{\sqrt{a}} < \infty$ can happen. Indeed, if $r_1 = s_1 = t = 1$, we have the following values: $b(2) = 6, b(9) = 15, b(20) = 28, b(54) = 66$. It is left to the reader to explain these values and show that they imply $\liminf_{a\to\infty} \frac{b(a)-a}{\sqrt{2a}} \leq 1$. In fact, much more should be true; I'm tempted to conjecture that $\log(a)$ is the right order of magnitude here. That is: $\liminf_{a\to\infty} \frac{b(a)-a}{\log(a)} = c$, for some $0 < c < \infty$. In a similar vein, it can be conjectured that $b(a-1) > b(a)$ happens infinitely often, which, at least when $r_1 = s_1 = t = 1$, shouldn't be too hard.

On another note, even in the classical special case of $r_1 = s_1 = t = 1$, questions remain. For example, it is still open if $\gcd(X_n, L_n) = 1$ holds for infinitely many n or not. This all boils down to how many l with $1 \leq l \leq p-1$ there exist for which $\sum_{i=1}^{l} i^{-1} \equiv 0 \pmod{p}$. A decent upper bound on that number ought to resolve this question.

At last, let, for simplicity, $s_i = 1$ for all i. Ernie Croot then asked: what happens when we just assume that there exists a finite set A, consisting of non-negative integers, such that $r_i \in A$, for all i? As it turns out, then it *is* possible that the denominator of the corresponding sum grows monotonically. More precisely, we shall sketch a proof by induction that if $r_i \in \{-1, 1\}$, then it is possible that the denominator of $\sum_{i=1}^{n} \frac{r_i}{i}$ is never smaller than $L_n = \text{lcm}\{1, 2, .., n\}$. Recall that $X_n = L_n \sum_{i=1}^{n} \frac{r_i}{i}$ and note that what we need to prove is equivalent to the assertion that $X_n \not\equiv 0 \pmod{p}$ for all prime p with $p \leq n$. For $n = 1$ this is trivial, so assume that for some $n \geq 2$, we have, for all $p < n$, that $X_{n-1} \not\equiv 0 \pmod{p}$.

If $n = lp^k$ implies $l > p$, then it doesn't matter whether $r_n = 1$ or $r_n = -1$. Because we then have: $X_n = X_{n-1} \pm \frac{L_n}{n}$, and $\frac{L_n}{n}$ vanishes modulo p, for all $p < n$.

If $n = lp^k$ for some l with $1 < l < p$, then $\frac{L_n}{n}$ still vanishes modulo p' for all $p' < n$ different from p, but doesn't vanish modulo p. But, $\frac{L_n}{n} \neq \frac{-L_n}{n} \pmod{p}$, since $p \neq 2$ (otherwise $l = 1$, contrary to our assumption). So we have that $X_{n-1} + \frac{L_n}{n} \neq X_{n-1} - \frac{L_n}{n} \pmod{p}$. So either $X_{n-1} + \frac{L_n}{n} \neq 0 \pmod{p}$, in which case we choose $r_n = 1$, or $X_{n-1} - \frac{L_n}{n} \neq 0 \pmod{p}$, in which case we choose $r_n = -1$.

If $n = p^k$, then $X_n = pX_{n-1} \pm \frac{L_n}{n} \equiv pX_{n-1} \pmod{p'} \neq 0 \pmod{p'}$, for all $p' < n$ different from p. And $X_n = pX_{n-1} \pm \frac{L_n}{n} \equiv \pm\frac{L_n}{n} \neq 0 \pmod{p}$. So in this case it also doesn't matter whether $r_n = 1$ or $r_n = -1$.

This method can also be used to show that if $r_i \in A$, then it is possible that the denominator of $\sum_{i=1}^{n} \frac{r_i}{i}$ is never smaller than $\text{lcm}\{1, 2, .., n\}$, as long as A contains an odd integer and if for every odd prime p, there exists $a_1, a_2 \in A$, such that $a_1 \neq a_2 \pmod{p}$. It is a fun exercise to prove that these conditions are also necessary.

4. Acknowledgements

I would like to thank Ernie Croot for the encouragements to obtain more general results, Ronald Graham for the encouragements to publish and Peter Kuijpers for his computer skills which helped shape my intuition.

References

[1] P. Erdös, R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory.* Enseign. Math. (2), vol. 28, Enseignement Math., Geneva, 1980. Also available here