SEQUENCES WITH BOUNDED LCM FOR CONSECUTIVE ELEMENTS

Wouter van Doorn
Undergraduate at Utrecht University, Utrecht, The Netherlands
wonterman1@hotmail.com

Abstract

Let $1 \leq a_1 < a_2 < .. < a_k \leq n$ be a sequence of positive integers, such that $\operatorname{lcm}(a_i, a_{i+1}) \leq n$ for all i with $1 \leq i \leq k-1$. In [1, p. 34], it is conjectured that $k = O(\sqrt{n})$. In this short note, we will provide a proof of this conjecture.

1. Main result and proof

We can immediately state our main Theorem:

Theorem 1. *$k < c\sqrt{n} + \log(2n)$, where $c = \sum_{j=1}^{\infty} \frac{1}{(j+1)\sqrt{j}} \approx 1.86$*

Proof. For $n \leq 4$, this is clear, so let's assume $n \geq 5$. Define B_j to be $\max(a_i : a_i - a_{i-1} \leq j)$, if this exists and 0 otherwise. Note that $B_n = a_k \leq n$. We have the following upper bound on k, in terms of the B_j:

$$
\begin{aligned}
k &\leq \sum_{j=1}^{n} \frac{B_j - B_{j-1}}{j} \\
&= \frac{B_n}{n} + \sum_{j=1}^{n-1} \frac{B_j}{j(j+1)} \\
&\leq 1 + \sum_{j=1}^{n-1} \frac{B_j}{j(j+1)}
\end{aligned}
$$

On the other hand, we also have an upper bound on B_j; if $a_{i-1} \geq \sqrt{jn}$, then:

$$
\begin{aligned}
n &\geq \operatorname{lcm}(a_i, a_{i+1}) \\
&= \frac{a_i * a_{i-1}}{\gcd(a_i, a_{i-1})} \\
&> \frac{a_{i-1}^2}{a_i - a_{i-1}} \\
&\geq \frac{jn}{a_i - a_{i-1}}
\end{aligned}
$$

implying that $a_i - a_{i-1} > j$, and thus we must have that $B_j < \sqrt{jn} + j$. Using this estimate, we obtain:

$$
\begin{aligned}
k &< 1 + \sum_{j=1}^{n-1} \frac{\sqrt{jn} + j}{j(j+1)} \\
&= \sqrt{n} \sum_{j=1}^{n-1} \frac{1}{(j+1)\sqrt{j}} + \sum_{j=1}^{n} \frac{1}{j} \\
&< c\sqrt{n} + \log(2n)
\end{aligned}
$$

And this finishes our proof.

□

References

[1] P. Erdös, R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*. Enseign. Math. (2), vol. 28, Enseignement Math., Geneva, 1980. Also available here.